

CHAPTER 17 - FOURIER TRANSFORM

List of topics for this chapter :
Fourier Transform and its Properties
Circuit Applications
Parseval's Theorem
Applications

FOURIER TRANSFORM AND ITS PROPERTIES

Problem 17.1 Find the Fourier Transform of the pulse shown in Figure 17.1.

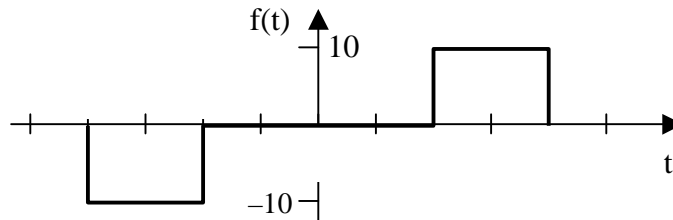


Figure 17.1

We begin with the derivative of $f(t)$.

$$\frac{df(t)}{dt} = -\delta(t+2) + \delta(t+1) + \delta(t-1) - \delta(t-2)$$

Transforming this into the frequency domain yields,

$$j\omega F(\omega) = -e^{j2\omega} + e^{j\omega} + e^{-j\omega} - e^{-j2\omega} = 2\cos(2\omega) - 2\cos(\omega)$$

Therefore,

$$F(\omega) = \frac{2(\cos(2\omega) - \cos(\omega))}{j\omega}$$

Problem 17.2 Find the inverse Fourier transforms of the following,

- (a) $10/[(j\omega)(j\omega + 5)]$
- (b) $5j\omega/[(-j\omega + 1)(j\omega + 2)]$
- (c) $(2 - j\omega)/(-\omega^2 + 4j\omega + 3)$
- (d) $3\delta(\omega)/[(j\omega + 2)(j\omega + 3)]$

Now to find the inverse transforms.

$$(a) \quad F(s) = \frac{10}{s(s+5)} = \frac{A}{s} + \frac{B}{s+5}, \quad A = 10/5 = 2 \text{ and } B = 10/-5 = -2$$

Therefore, $F(j\omega) = (2/j\omega) - (2/(j\omega + 5))$

Transforming, $f(t) = \underline{\mathbf{sgn(t)} - 2e^{-5t}u(t)}$

$$(b) \quad F(s) = \frac{5s}{(-s+1)(s+2)} = \frac{-5s}{(s-1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+2}$$

$$A = -5/(1+2) = -5/3 \text{ and } B = -5x(-2)/(-2-1) = -10/3$$

Therefore, $F(j\omega) = [(-5/3)/(j\omega - 1)] + [(-10/3)/(j\omega + 2)]$

Transforming, $f(t) = \underline{\mathbf{-\frac{5}{3}e^t u(-t) - \frac{10}{3}e^{-2t}u(t)}}$

$$(c) \quad F(s) = \frac{(2-s)}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3}, \quad A = 3/2 \text{ and } B = -5/2$$

Therefore, $F(j\omega) = [1.5/(j\omega + 1)] - [2.5/(j\omega + 3)]$

Transforming, $f(t) = \underline{\mathbf{1.5e^{-t}u(t) - 2.5e^{-3t}u(t)}}$

$$(d) \quad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{3\delta(\omega)e^{j\omega t}}{(j\omega+2)(j\omega+3)} dt = \frac{1}{2\pi} \frac{3}{6} = \underline{\mathbf{\frac{1}{4\pi}}}$$

Problem 17.3 [17.7] Find the Fourier transform of the "sine-wave pulse" shown in Figure 17.1.

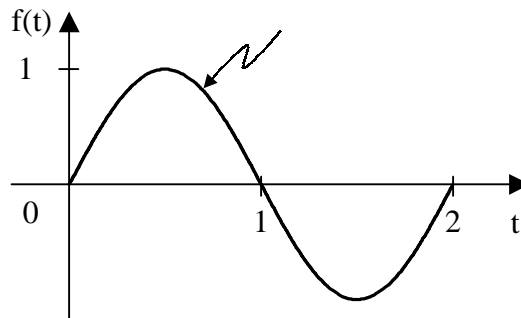


Figure 17.1

$$f(t) = \sin(\pi t)[u(t) - u(t-2)]$$

$$F(\omega) = \int_0^2 \sin(\pi t) e^{-j\omega t} dt = \frac{1}{2j} \int_0^2 (e^{j\pi} - e^{-j\pi})(e^{-j\omega t}) dt$$

$$F(\omega) = \frac{1}{2j} \int_0^2 e^{j(-\omega+\pi)t} + e^{-j(\omega+\pi)t} dt$$

$$F(\omega) = \frac{1}{2j} \left[\frac{1}{-j(\omega-\pi)} e^{-j(\omega-\pi)t} \Big|_0^2 + \frac{1}{-j(\omega+\pi)} e^{-j(\omega+\pi)t} \Big|_0^2 \right]$$

$$F(\omega) = \frac{1}{2} \left(\frac{1 - e^{-j2\omega}}{\pi - \omega} + \frac{1 - e^{-j2\omega}}{\pi + \omega} \right)$$

$$F(\omega) = \frac{1}{(2)(\pi^2 - \omega^2)} (2\pi + 2\pi e^{-j2\omega})$$

$$F(\omega) = \frac{\pi}{\omega^2 - \pi^2} (e^{-j2\omega} - 1)$$

CIRCUIT APPLICATIONS

Problem 17.4 Find the transfer function, $V_o(\omega)/V_s(\omega)$ for the circuit shown in Figure 17.3.

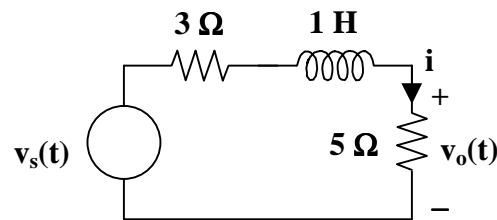


Figure 17.1

First we will solve for I .

$$I = \frac{V_s(\omega)}{3 + j\omega + 5} = \frac{V_s(\omega)}{j\omega + 8}, \text{ and } V_o(\omega) = 5I$$

Therefore,

$$\frac{V_o(\omega)}{V_s(\omega)} = \frac{5}{j\omega + 8}$$

Problem 17.5 Solve for $v_C(t)$ in Figure 17.4, where $i(t) = u(t)$ A.

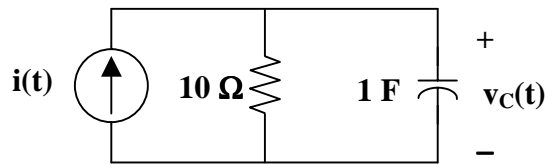


Figure 17.1

First we transform $i(t)$ into the frequency domain.

$$I(\omega) = \pi\delta(\omega) + 1/(j\omega), \text{ and } V_C(\omega) = I(\omega) \frac{10 \frac{1}{j\omega}}{10 + \frac{1}{j\omega}} = I(\omega) \frac{1}{j\omega + 0.1}$$

Therefore,
$$V_C(\omega) = \frac{\pi\delta(\omega)}{j\omega} + \frac{1}{j\omega(j\omega + 0.1)} = V_1 + V_2$$

$$V_2 = \frac{1}{s(s + 0.1)} = \frac{A}{s} + \frac{B}{s + 0.1}, \text{ where } A = 1/0.1 = 10 \text{ and } B = 1/(-0.1) = -10$$

Therefore,
$$v_2(t) = 5\text{sgn}(t) - 10e^{-t/10}u(t)$$

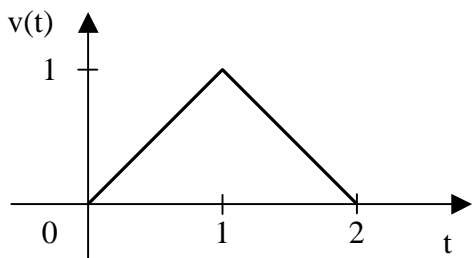
$$v_1(t) = \frac{1}{2\pi} \int \frac{\pi\delta(\omega)}{j\omega + 0.1} e^{j\omega t} d\omega = \frac{1}{2\pi} \frac{\pi}{0.1} = 5$$

This leads to
$$v_o(t) = 5 - 5\text{sgn}(t) - 10e^{-t/10}u(t), \text{ but } \text{sgn}(t) = -1 + 2u(t)$$

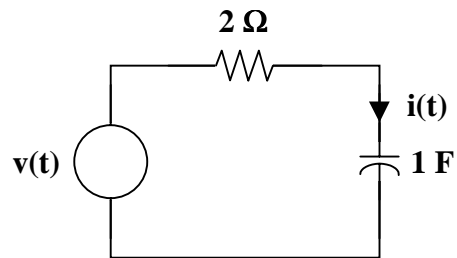
Therefore,
$$v_o(t) = 5 - 5 + 10u(t) - 10e^{-t/10}u(t)$$

or
$$v_o(t) = \underline{10(1 - e^{-t/10})u(t) \text{ volts}}$$

Problem 17.6 [17.29] Determine the current $i(t)$ in the circuit of Figure 17.1(b), given the voltage source shown in Figure 17.1(a).



(a)



(b)

Figure 17.1

$$\ddot{v}(t) = \delta(t) - 2\delta(t-1) + \delta(t-2)$$

$$-\omega^2 V(\omega) = 1 - 2e^{-j\omega} + e^{j\omega 2}$$

$$V(\omega) = \frac{1 - 2e^{-j\omega} + e^{j\omega 2}}{-\omega^2}$$

Now, $Z(\omega) = 2 + \frac{1}{j\omega} = \frac{1 + j2\omega}{j\omega}$

$$I = \frac{V(\omega)}{Z(\omega)} = \frac{2e^{j\omega} - e^{j\omega 2} - 1}{\omega^2} \cdot \frac{j\omega}{1 + j2\omega}$$

$$I = \frac{1}{(j\omega)(0.5 + j\omega)} (0.5 + 0.5e^{-j\omega 2} - e^{-j\omega})$$

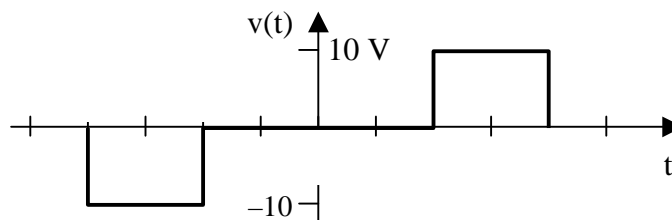
But $\frac{1}{(s)(s+0.5)} = \frac{A}{s} + \frac{B}{s+0.5} \longrightarrow A = 2, B = -2$

$$I(\omega) = \left(\frac{2}{j\omega}\right)(0.5 + 0.5e^{j\omega 2} - e^{-j\omega}) - \left(\frac{2}{0.5 + j\omega}\right)(0.5 + 0.5e^{-j\omega 2} - e^{-j\omega})$$

$$i(t) = \frac{1}{2} \text{sgn}(t) + \frac{1}{2} \text{sgn}(t-2) - \text{sgn}(t-1) - e^{-0.5t} \mathbf{u}(t) - e^{-0.5(t-2)} \mathbf{u}(t-2) - 2e^{-0.5(t-1)} \mathbf{u}(t-1)$$

PARSEVAL'S THEOREM

Problem 17.7 Find the total energy in $v(t)$ where $v(t)$ is the pulse shown below.



In the time domain, $W_{1\Omega} = \int_{-2}^{-1} (-10)^2 dt + \int_1^2 10^2 dt = 100t \Big|_{-2}^{-1} + 100t \Big|_1^2$

$$= -100 + 200 + 200 - 100 = \underline{\underline{200 \text{ J}}}$$

Problem 17.8 [17.43] A voltage source $v_s(t) = e^{-t} \sin(2t)u(t)$ V is applied to a 1- Ω resistor. Calculate the energy delivered to the resistor.

$$W_{1\Omega} = \int_{-\infty}^{\infty} f^2(t) dt = \int_0^{\infty} e^{-2t} \sin^2(2t) dt$$

But $\sin^2(A) = \frac{1}{2} [1 - \cos(2A)]$

$$W_{1\Omega} = \int_0^{\infty} e^{-2t} (0.5)[1 - \cos(4t)] dt = \frac{1}{2} \cdot \frac{e^{-2t}}{-2} \Big|_0^{\infty} - \frac{e^{-2t}}{4+16} [-2\cos(4t) + 4\sin(4t)]_0^{\infty}$$

$$W_{1\Omega} = \left(\frac{1}{4}\right) + \left(\frac{1}{20}\right)(-2) = \underline{\underline{0.15 \text{ J}}}$$

APPLICATIONS

Problem 17.9 Given the AM signal,

$$f(t) = 10(1 + 4\cos(2000\pi t))\cos(\pi \times 10^6 t),$$

solve for the:

- (a) the carrier frequency
- (b) the lower sideband frequency
- (c) the upper sideband frequency

$$\omega_m = 2000\pi = 2\pi f \text{ which leads to } f = 1 \text{ kHz}$$

(a) $\omega_c = \pi \times 10^6 = 2\pi f_c$ which leads to $f_c = \underline{\underline{500 \text{ kHz or } 0.5 \text{ MHz}}}$

(b) $Lsb = f_c - f_m = (500 - 1) \text{ kHz} = 499 \text{ kHz}$

(c) $Usb = f_c + f_m = (500 + 1) \text{ kHz} = 501 \text{ kHz}$

Problem 17.10 [17.47] A voice signal occupying the frequency band of 0.4 to 3.5 kHz is used to amplitude modulate a 10-MHz carrier. Determine the range of frequencies for the lower and upper sidebands.

For the lower sideband, the frequencies range from

$$10,000,000 - 3,500 = \underline{\mathbf{9,996,500 \text{ Hz}}}$$

to

$$10,000,000 - 400 = \underline{\mathbf{9,999,600 \text{ Hz}}}$$

For the upper sideband, the frequencies range from

$$10,000,000 + 400 = \underline{\mathbf{10,000,400 \text{ Hz}}}$$

to

$$10,000,000 + 3,500 = \underline{\mathbf{10,003,500 \text{ Hz}}}$$