14

Radiation Fields

14.1 Currents and Charges as Sources of Fields

Here we discuss how a given distribution of currents and charges can generate and radiate electromagnetic waves. Typically, the current distribution is localized in some region of space (for example, currents on a wire antenna.) The current source generates electromagnetic fields, which can propagate to far distances from the source location.

It proves convenient to work with the electric and magnetic *potentials* rather than the $\boldsymbol{E}$ and $\boldsymbol{H}$ fields themselves. Basically, two of Maxwell's equations allow us to introduce these potentials; then, the other two, written in terms of these potentials, take a simple wave-equation form. The two Maxwell equations,

$$\nabla\cdot\boldsymbol{B}=0,\quad \nabla\times\boldsymbol{E}=-\frac{\partial\boldsymbol{B}}{\partial t} \tag{14.1.1}$$

imply the existence of the magnetic and electric potentials $\boldsymbol{A}(\mathbf{r},t)$ and $\varphi(\mathbf{r},t)$, such that the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are obtainable by

$$\boxed{\begin{aligned}&\boldsymbol{E}=-\nabla\varphi-\frac{\partial\boldsymbol{A}}{\partial t}\\&\boldsymbol{B}=\nabla\times\boldsymbol{A}\end{aligned}} \tag{14.1.2}$$

Indeed, the divergenceless of $\boldsymbol{B}$ implies the existence of $\boldsymbol{A}$, such that $\boldsymbol{B}=\nabla\times\boldsymbol{A}$. Then, Faraday's law can be written as

$$\nabla\times\boldsymbol{E}=-\frac{\partial\boldsymbol{B}}{\partial t}=-\nabla\times\frac{\partial\boldsymbol{A}}{\partial t}\quad\Rightarrow\quad\nabla\times\big(\boldsymbol{E}+\frac{\partial\boldsymbol{A}}{\partial t}\big)=0$$

Thus, the quantity $\boldsymbol{E}+\partial\boldsymbol{A}/\partial t$ is curl-less and can be represented as the gradient of a scalar potential, that is, $\boldsymbol{E}+\partial\boldsymbol{A}/\partial t=-\nabla\varphi$.

The potentials $\boldsymbol{A}$ and φ are not uniquely defined. For example, they may be changed by adding constants to them. Even more freedom is possible, known as *gauge invariance* of Maxwell's equations. Indeed, for any scalar function $f(\mathbf{r},t)$, the following *gauge transformation* leaves $\boldsymbol{E}$ and $\boldsymbol{B}$ invariant:

$$\boxed{\begin{aligned}\varphi' &= \varphi - \frac{\partial f}{\partial t} \\ \boldsymbol{A}' &= \boldsymbol{A} + \nabla f\end{aligned}} \qquad \text{(gauge transformation)} \tag{14.1.3}$$

For example, we have for the electric field:

$$\boldsymbol{E}' = -\nabla\varphi' - \frac{\partial \boldsymbol{A}'}{\partial t} = -\nabla\big(\varphi - \frac{\partial f}{\partial t}\big) - \frac{\partial}{\partial t}(\boldsymbol{A} + \nabla f) = -\nabla\varphi - \frac{\partial \boldsymbol{A}}{\partial t} = \boldsymbol{E}$$

This freedom in selecting the potentials allows us to impose some convenient *constraints* between them. In discussing radiation problems, it is customary to impose the *Lorenz condition*:†

$$\boxed{\nabla\cdot\boldsymbol{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t} = 0} \qquad \text{(Lorenz condition)} \tag{14.1.4}$$

We will also refer to it as *Lorenz gauge or radiation gauge.* Under the gauge transformation (14.1.3), we have:

$$\nabla\cdot\boldsymbol{A}' + \frac{1}{c^2}\frac{\partial\varphi'}{\partial t} = \big(\nabla\cdot\boldsymbol{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t}\big) - \big(\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} - \nabla^2 f\big)$$

Therefore, if $\boldsymbol{A},\varphi$ did not satisfy the constraint (14.1.4), the transformed potentials $\boldsymbol{A}',\varphi'$ could be made to satisfy it by an appropriate choice of the function f, that is, by choosing f to be the solution of the inhomogeneous wave equation:

$$\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} - \nabla^2 f = \nabla\cdot\boldsymbol{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t}$$

Using Eqs. (14.1.2) and (14.1.4) into the remaining two of Maxwell's equations,

$$\nabla\cdot\boldsymbol{E} = \frac{1}{\epsilon}\rho, \quad \nabla\times\boldsymbol{B} = \mu\boldsymbol{J} + \frac{1}{c^2}\frac{\partial\boldsymbol{E}}{\partial t} \tag{14.1.5}$$

we find,

$$\begin{aligned}\nabla\cdot\boldsymbol{E} &= \nabla\cdot\big(-\nabla\varphi - \frac{\partial\boldsymbol{A}}{\partial t}\big) = -\nabla^2\varphi - \frac{\partial}{\partial t}(\nabla\cdot\boldsymbol{A}) = -\nabla^2\varphi - \frac{\partial}{\partial t}\big(-\frac{1}{c^2}\frac{\partial\varphi}{\partial t}\big) \\ &= \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \nabla^2\varphi\end{aligned}$$

and, similarly,

$$\begin{aligned}\nabla\times\boldsymbol{B} - \frac{1}{c^2}\frac{\partial\boldsymbol{E}}{\partial t} &= \nabla\times(\nabla\times\boldsymbol{A}) - \frac{1}{c^2}\frac{\partial}{\partial t}\big(-\nabla\varphi - \frac{\partial\boldsymbol{A}}{\partial t}\big) \\ &= \nabla\times(\nabla\times\boldsymbol{A}) + \nabla\big(\frac{1}{c^2}\frac{\partial\varphi}{\partial t}\big) + \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2}\end{aligned}$$

†Almost universally wrongly attributed to H. A. Lorentz instead of L. V. Lorenz. See Refs. [64–70] for the historical roots of scalar and vector potentials and gauge transformations.

$$\begin{aligned}&= \nabla\times(\nabla\times\boldsymbol{A}) - \nabla(\nabla\cdot\boldsymbol{A}) + \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} \\ &= \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} - \nabla^2\boldsymbol{A}\end{aligned}$$

where we used the identity $\nabla\times(\nabla\times\boldsymbol{A}) = \nabla(\nabla\cdot\boldsymbol{A}) - \nabla^2\boldsymbol{A}$. Therefore, Maxwell's equations (14.1.5) take the equivalent wave-equation forms for the potentials:

$$\boxed{\begin{aligned}\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \nabla^2\varphi &= \frac{1}{\epsilon}\rho \\ \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} - \nabla^2\boldsymbol{A} &= \mu\boldsymbol{J}\end{aligned}} \qquad \text{(wave equations)} \tag{14.1.7}$$

To summarize, the densities $\rho,\boldsymbol{J}$ may be thought of as the *sources* that generate the potentials $\varphi,\boldsymbol{A}$, from which the fields $\boldsymbol{E},\boldsymbol{B}$ may be computed via Eqs. (14.1.2).

The Lorenz condition is compatible with Eqs. (14.1.7) and implies charge conservation. Indeed, we have from (14.1.7)

$$\big(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\big)\big(\nabla\cdot\boldsymbol{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t}\big) = \mu\nabla\cdot\boldsymbol{J} + \frac{1}{\epsilon c^2}\frac{\partial\rho}{\partial t} = \mu\big(\nabla\cdot\boldsymbol{J} + \frac{\partial\rho}{\partial t}\big)$$

where we used $\mu\epsilon = 1/c^2$. Thus, the Lorenz condition (14.1.4) implies the charge conservation law:

$$\nabla\cdot\boldsymbol{J} + \frac{\partial\rho}{\partial t} = 0 \tag{14.1.8}$$

14.2 Retarded Potentials

The main result that we would like to show here is that if the source densities $\rho,\boldsymbol{J}$ are known, the causal solutions of the wave equations (14.1.7) are given by:

$$\boxed{\begin{aligned}\varphi(\mathbf{r},t) &= \int_V \frac{\rho(\mathbf{r}',t-\frac{R}{c})}{4\pi\epsilon R}\,d^3\mathbf{r}' \\ \boldsymbol{A}(\mathbf{r},t) &= \int_V \frac{\mu\boldsymbol{J}(\mathbf{r}',t-\frac{R}{c})}{4\pi R}\,d^3\mathbf{r}'\end{aligned}} \qquad \text{(retarded potentials)} \tag{14.2.1}$$

where $R = |\mathbf{r}-\mathbf{r}'|$ is the distance from the field (observation) point $\mathbf{r}$ to the source point $\mathbf{r}'$, as shown in Fig. 14.2.1. The integrations are over the localized volume V in which the source densities $\rho,\boldsymbol{J}$ are non-zero.

In words, the potential $\varphi(\mathbf{r},t)$ at a field point $\mathbf{r}$ at time t is obtainable by superimposing the fields due to the infinitesimal charge $\rho(\mathbf{r}',t')d^3\mathbf{r}'$ that resided within the volume element $d^3\mathbf{r}'$ at time instant t', which is R/c seconds earlier than t, that is, $t' = t - R/c$.

Thus, in accordance with our intuitive notions of causality, a change at the source point $\mathbf{r}'$ is not felt instantaneously at the field point $\mathbf{r}$, but takes R/c seconds to get there, that is, it propagates with the speed of light. Equations (14.2.1) are referred to

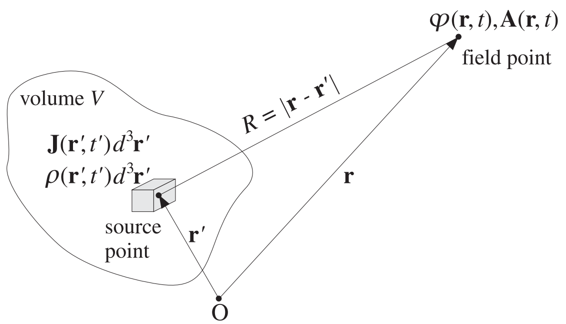


Fig. 14.2.1 Retarded potentials generated by a localized current/charge distribution.

as the *retarded* potentials because the sources inside the integrals are evaluated at the retarded time $t' = t - R/c$.

To prove (14.2.1), we consider first the solution to the following scalar wave equation driven by a time-dependent point source located at the origin:

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \nabla^2 u = f(t)\,\delta^{(3)}(\mathbf{r}) \tag{14.2.2}$$

where $f(t)$ is an arbitrary function of time and $\delta^{(3)}(\mathbf{r})$ is the 3-dimensional delta function. We show below that the causal solution of Eq. (14.2.2) is:†

$$u(\mathbf{r},t) = \frac{f(t')}{4\pi r} = \frac{f\big(t - \frac{r}{c}\big)}{4\pi r} = f\big(t - \frac{r}{c}\big)\,g(\mathbf{r})\,, \qquad \text{where} \quad g(\mathbf{r}) = \frac{1}{4\pi r} \tag{14.2.3}$$

with $t' = t - r/c$ and $r = |\mathbf{r}|$. The function $g(\mathbf{r})$ is recognized as the Green's function for the electrostatic Coulomb problem and satisfies:

$$\boldsymbol{\nabla} g = -\hat{\mathbf{r}}\,\frac{1}{4\pi r^2} = -\hat{\mathbf{r}}\,\frac{g}{r}\,, \qquad \nabla^2 g = -\delta^{(3)}(\mathbf{r}) \tag{14.2.4}$$

where $\hat{\mathbf{r}} = \mathbf{r}/r$ is the radial unit vector. We note also that because $f(t - r/c)$ depends on r only through its t-dependence, we have:

$$\frac{\partial}{\partial r} f(t - r/c) = -\frac{1}{c}\frac{\partial}{\partial t} f(t - r/c) = -\frac{1}{c}\dot{f}$$

It follows that $\boldsymbol{\nabla} f = -\hat{\mathbf{r}}\dot{f}/c$ and

$$\nabla^2 f = -(\boldsymbol{\nabla}\cdot\hat{\mathbf{r}})\frac{\dot{f}}{c} - \frac{1}{c}\hat{\mathbf{r}}\cdot\boldsymbol{\nabla}\dot{f} = -(\boldsymbol{\nabla}\cdot\hat{\mathbf{r}})\frac{\dot{f}}{c} - \frac{1}{c}\hat{\mathbf{r}}\cdot\Big(-\hat{\mathbf{r}}\frac{\ddot{f}}{c}\Big) = -\frac{2\dot{f}}{cr} + \frac{1}{c^2}\ddot{f} \tag{14.2.5}$$

where we used the result $\boldsymbol{\nabla}\cdot\hat{\mathbf{r}} = 2/r$.‡ Using Eqs. (14.2.3)–(14.2.5) into the identity:

$$\nabla^2 u = \nabla^2(fg) = 2\boldsymbol{\nabla} f\cdot\boldsymbol{\nabla} g + g\nabla^2 f + f\nabla^2 g$$

†The anticausal, or time-advanced, solution is $u(\mathbf{r},t) = f(t + r/c)\,g(\mathbf{r})$.

‡Indeed, $\boldsymbol{\nabla}\cdot\hat{\mathbf{r}} = \boldsymbol{\nabla}\cdot(\mathbf{r}/r) = (\boldsymbol{\nabla}\cdot\mathbf{r})/r + \mathbf{r}\cdot(-\hat{\mathbf{r}}/r^2) = 3/r - 1/r = 2/r$.

we obtain,

$$\nabla^2 u = 2\Big(-\hat{\mathbf{r}}\frac{\dot{f}}{c}\Big)\cdot\Big(-\hat{\mathbf{r}}\frac{g}{r}\Big) - \frac{2\dot{f}}{cr}g + \frac{1}{c^2}\ddot{f}g - f\big(t - \frac{r}{c}\big)\,\delta^{(3)}(\mathbf{r})$$

The first two terms cancel and the fourth term can be written as $f(t)\,\delta^{(3)}(\mathbf{r})$ because the delta function forces $\mathbf{r} = 0$. Recognizing that the third term is

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{1}{c^2}\ddot{f}g$$

we have,

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - f(t)\,\delta^{(3)}(\mathbf{r})$$

which shows Eq. (14.2.2). Next, we shift the point source to location $\mathbf{r}'$, and find the solution to the wave equation:

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \nabla^2 u = f(\mathbf{r}',t)\,\delta^{(3)}(\mathbf{r}-\mathbf{r}') \quad\Rightarrow\quad u(\mathbf{r},t) = \frac{f(\mathbf{r}', t - R/c)}{4\pi R} \tag{14.2.6}$$

where $R = |\mathbf{r}-\mathbf{r}'|$ and we have allowed the function f to also depend on $\mathbf{r}'$. Note that here $\mathbf{r}'$ is fixed and the field point $\mathbf{r}$ is variable.

Using linearity, we may form now the linear combination of several such point sources located at various values of $\mathbf{r}'$ and get the corresponding linear combination of solutions. For example, the sum of two sources will result in the sum of solutions:

$$f(\mathbf{r}_1',t)\,\delta^{(3)}(\mathbf{r}-\mathbf{r}_1') + f(\mathbf{r}_2',t)\,\delta^{(3)}(\mathbf{r}-\mathbf{r}_2') \quad\Rightarrow\quad \frac{f(\mathbf{r}_1', t - R_1/c)}{4\pi R_1} + \frac{f(\mathbf{r}_2', t - R_2/c)}{4\pi R_2}$$

where $R_1 = |\mathbf{r}-\mathbf{r}_1'|$, $R_2 = |\mathbf{r}-\mathbf{r}_2'|$. More generally, integrating over the whole volume V over which $f(\mathbf{r}',t)$ is nonzero, we have for the sum of sources:

$$f(\mathbf{r},t) = \int_V f(\mathbf{r}',t)\,\delta^{(3)}(\mathbf{r}-\mathbf{r}')\,d^3\mathbf{r}'$$

and the corresponding sum of solutions:

$$u(\mathbf{r},t) = \int_V \frac{f(\mathbf{r}', t - R/c)}{4\pi R}\,d^3\mathbf{r}' \tag{14.2.7}$$

where $R = |\mathbf{r}-\mathbf{r}'|$. Thus, this is the causal solution to the general wave equation:

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \nabla^2 u = f(\mathbf{r},t) \tag{14.2.8}$$

The retarded potentials (14.2.1) are special cases of Eq. (14.2.7), applied for $f(\mathbf{r},t) = \rho(\mathbf{r},t)/\epsilon$ and $f(\mathbf{r},t) = \mu\mathbf{J}(\mathbf{r},t)$.

14.3 Harmonic Time Dependence

Since we are primarily interested in single-frequency waves, we will Fourier transform all previous results. This is equivalent to assuming a sinusoidal time dependence $e^{j\omega t}$ for all quantities. For example,

$$\varphi(\mathbf{r},t)=\varphi(\mathbf{r})e^{j\omega t},\quad \rho(\mathbf{r},t)=\rho(\mathbf{r})e^{j\omega t},\quad \text{etc.}$$

Then, the retarded solutions (14.2.1) become:

$$\varphi(\mathbf{r})e^{j\omega t}=\int_V \frac{\rho(\mathbf{r}')e^{j\omega(t-\frac{R}{c})}}{4\pi\epsilon R}\,d^3\mathbf{r}'$$

Canceling a common factor $e^{j\omega t}$ from both sides, we obtain for the phasor part of the retarded potentials, where $R=|\mathbf{r}-\mathbf{r}'|$:

$$\boxed{\begin{aligned}\varphi(\mathbf{r})&=\int_V \frac{\rho(\mathbf{r}')e^{-jkR}}{4\pi\epsilon R}\,d^3\mathbf{r}'\\ \boldsymbol{A}(\mathbf{r})&=\int_V \frac{\mu\boldsymbol{J}(\mathbf{r}')e^{-jkR}}{4\pi R}\,d^3\mathbf{r}'\end{aligned}},\qquad \text{where}\quad \boxed{k=\frac{\omega}{c}} \tag{14.3.1}$$

The quantity k represents the free-space wavenumber and is related to the wavelength via $k=2\pi/\lambda$. An alternative way to obtain Eqs. (14.3.1) is to start with the wave equations and replace the time derivatives by $\partial_t\to j\omega$. Equations (14.1.7) become then the Helmholtz equations:

$$\boxed{\begin{aligned}\nabla^2\varphi+k^2\varphi&=-\frac{1}{\epsilon}\rho\\ \nabla^2\boldsymbol{A}+k^2\boldsymbol{A}&=-\mu\boldsymbol{J}\end{aligned}} \tag{14.3.2}$$

Their solutions may be written in the convolutional form:†

$$\boxed{\begin{aligned}\varphi(\mathbf{r})&=\int_V \frac{1}{\epsilon}\rho(\mathbf{r}')G(\mathbf{r}-\mathbf{r}')\,d^3\mathbf{r}'\\ \boldsymbol{A}(\mathbf{r})&=\int_V \mu\boldsymbol{J}(\mathbf{r}')G(\mathbf{r}-\mathbf{r}')\,d^3\mathbf{r}'\end{aligned}} \tag{14.3.3}$$

where $G(\mathbf{r})$ is the Green's function for the Helmholtz equation:

$$\nabla^2G+k^2G=-\delta^{(3)}(\mathbf{r}),\qquad G(\mathbf{r})=\frac{e^{-jkr}}{4\pi r} \tag{14.3.4}$$

Replacing $\partial/\partial t$ by $j\omega$, the Lorenz condition (14.1.4) takes the form:

$$\nabla\cdot\boldsymbol{A}+j\omega\mu\epsilon\varphi=0 \tag{14.3.5}$$

†The integrals in (14.3.1) or (14.3.3) are *principal-value* integrals, that is, the limits as $\delta\to 0$ of the integrals over $V-V_\delta(\mathbf{r})$, where $V_\delta(\mathbf{r})$ is an excluded small sphere of radius δ centered about $\mathbf{r}$. See Appendix D and Refs. [1153,460,472,598] and [106–110] for the properties of such principal value integrals.

Similarly, the electric and magnetic fields (14.1.2) become:

$$\boxed{\begin{aligned}\boldsymbol{E}&=-\nabla\varphi-j\omega\boldsymbol{A}\\ \boldsymbol{H}&=\frac{1}{\mu}\nabla\times\boldsymbol{A}\end{aligned}} \tag{14.3.6}$$

With the help of the Lorenz condition the $\boldsymbol{E}$-field can be expressed completely in terms of the vector potential. Solving (14.3.5) for the scalar potential, $\varphi=-\nabla\cdot\boldsymbol{A}/j\omega\mu\epsilon$, and substituting in (14.3.6), we find

$$\boldsymbol{E}=\frac{1}{j\omega\mu\epsilon}\nabla(\nabla\cdot\boldsymbol{A})-j\omega\boldsymbol{A}=\frac{1}{j\omega\mu\epsilon}\big[\nabla(\nabla\cdot\boldsymbol{A})+k^2\boldsymbol{A}\big]$$

where we used $\omega^2\mu\epsilon=\omega^2/c^2=k^2$. To summarize, with $\boldsymbol{A}(\mathbf{r})$ computed from Eq. (14.3.1), the $\boldsymbol{E},\boldsymbol{H}$ fields are obtained from:

$$\boxed{\begin{aligned}\boldsymbol{E}&=\frac{1}{j\omega\mu\epsilon}\big[\nabla(\nabla\cdot\boldsymbol{A})+k^2\boldsymbol{A}\big]\\ \boldsymbol{H}&=\frac{1}{\mu}\nabla\times\boldsymbol{A}\end{aligned}} \tag{14.3.7}$$

An alternative way of expressing the electric field is:

$$\boxed{\boldsymbol{E}=\frac{1}{j\omega\mu\epsilon}\big[\nabla\times(\nabla\times\boldsymbol{A})-\mu\boldsymbol{J}\big]} \tag{14.3.8}$$

This is Ampère's law solved for $\boldsymbol{E}$. When applied to a *source-free* region of space, such as in the radiation zone, (14.3.8) simplifies into:

$$\boxed{\boldsymbol{E}=\frac{1}{j\omega\mu\epsilon}\nabla\times(\nabla\times\boldsymbol{A})} \tag{14.3.9}$$

The fields $\boldsymbol{E},\boldsymbol{H}$ can also be expressed directly in terms of the sources $\rho,\boldsymbol{J}$. Indeed, replacing the solutions (14.3.3) into Eqs. (14.3.6) or (14.3.7), we obtain:

$$\boxed{\begin{aligned}\boldsymbol{E}&=\int_V\Big[-j\omega\mu\boldsymbol{J}G+\frac{1}{\epsilon}\rho\nabla'G\Big]dV'=\frac{1}{j\omega\epsilon}\int_V\big[(\boldsymbol{J}\cdot\nabla')\nabla'G+k^2\boldsymbol{J}G\big]dV'\\ \boldsymbol{H}&=\int_V\boldsymbol{J}\times\nabla'G\,dV'\end{aligned}} \tag{14.3.10}$$

Here, $\rho,\boldsymbol{J}$ stand for $\rho(\mathbf{r}'),\boldsymbol{J}(\mathbf{r}')$. The gradient operator ∇ acts inside the integrands only on G and because that depends on the difference $\mathbf{r}-\mathbf{r}'$, we can replace the gradient with $\nabla G(\mathbf{r}-\mathbf{r}')=-\nabla'G(\mathbf{r}-\mathbf{r}')$. Also, we denoted $d^3\mathbf{r}'$ by dV'.

In obtaining (14.3.10), we had to interchange the operator ∇ and the integrals over V. When $\mathbf{r}$ is outside the volume V—as is the case for most of our applications—then, such interchanges are valid. When $\mathbf{r}$ lies within V, then, interchanging single ∇'s is still valid, as in the first expression for $\boldsymbol{E}$ and for $\boldsymbol{H}$. However, in interchanging double ∇'s, additional source terms arise. For example, using Eq. (D.8) of Appendix D, we find by interchanging the operator $\nabla\times\nabla\times$ with the integral for $\boldsymbol{A}$ in Eq. (14.3.8):

$$\boldsymbol{E} = \frac{1}{j\omega\epsilon}\Big[\nabla\times\nabla\times\int_V \boldsymbol{J}G\,dV' - \boldsymbol{J}\Big] = \frac{1}{j\omega\epsilon}\Big[\frac{2}{3}\boldsymbol{J} + \mathrm{PV}\int_V \nabla\times\nabla\times(\boldsymbol{J}G)\,dV' - \boldsymbol{J}\Big]$$

where "PV" stands for "principal value." Because ∇ does not act on $\boldsymbol{J}(\mathbf{r}')$, we have:

$$\nabla\times\nabla\times(\boldsymbol{J}G) = \nabla\times(\nabla G\times\boldsymbol{J}) = (\boldsymbol{J}\cdot\nabla)\nabla G - \boldsymbol{J}\nabla^2 G = (\boldsymbol{J}\cdot\nabla')\nabla' G + k^2\boldsymbol{J}G$$

where in the last step, we replaced ∇ by $-\nabla'$ and $\nabla^2 G = -k^2 G$. It follows that:

$$\boldsymbol{E} = \frac{1}{j\omega\epsilon}\Big[\mathrm{PV}\int_V\big[(\boldsymbol{J}\cdot\nabla')\nabla' G + k^2\boldsymbol{J}G\big]dV' - \frac{1}{3}\boldsymbol{J}\Big], \quad (\mathbf{r}\text{ lies in }V) \tag{14.3.11}$$

In Sec. 17.10, we consider Eqs. (14.3.10) further in connection with Huygens's principle and vector diffraction theory.

Next, we present three illustrative applications of the techniques discussed in this section: (a) Determining the fields of linear wire antennas, (b) The fields produced by electric and magnetic dipoles, and (c) the Ewald-Oseen extinction theorem and the microscopic origin of the refractive index. Then, we go on in Sec. 14.7 to discuss the simplification of the retarded potentials (14.3.3) for radiation problems.

14.4 Fields of a Linear Wire Antenna

Eqs. (14.3.7) simplify considerably in the special practical case of a linear wire antenna, that is, a *thin* cylindrical antenna. Figure 14.4.1 shows the geometry in the case of a z-directed antenna of finite length with a current $I(z')$ flowing on it.

The assumption that the radius of the wire is much smaller than its length means effectively that the current density $\boldsymbol{J}(\mathbf{r}')$ will be z-directed and confined to zero transverse dimensions, that is,

$$\boldsymbol{J}(\mathbf{r}') = \hat{\mathbf{z}}\,I(z')\,\delta(x')\,\delta(y') \qquad \text{(current on thin wire antenna)} \tag{14.4.1}$$

In the more realistic case of an antenna of finite radius a, the current density will be confined to flow on the cylindrical surface of the antenna, that is, at radial distance $\rho = a$. Assuming cylindrical symmetry, the current density will be:

$$\boldsymbol{J}(\mathbf{r}') = \hat{\mathbf{z}}\,I(z')\,\delta(\rho' - a)\,\frac{1}{2\pi a} \tag{14.4.2}$$

This case is discussed in more detail in Chap. 21. In both cases, integrating the current density over the transverse dimensions of the antenna gives the current:

$$\int \boldsymbol{J}(x',y',z')\,dx'dy' = \int \boldsymbol{J}(\rho',\phi',z')\,\rho'\,d\rho'\,d\phi' = \hat{\mathbf{z}}\,I(z')$$

Because of the cylindrical symmetry of the problem, the use of cylindrical coordinates is appropriate, especially in determining the fields *near* the antenna (cylindrical coordinates are reviewed in Sec. 14.8.) On the other hand, that the radiated fields at

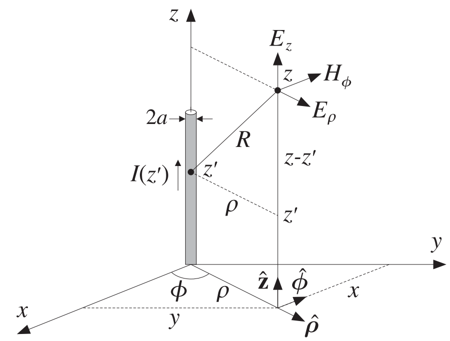


Fig. 14.4.1 Thin wire antenna.

far distances from the antenna are best described in spherical coordinates. This is so because any finite current source appears as a point from far distances.

Inserting Eq. (14.4.1) into Eq. (14.3.1), it follows that the vector potential will be z-directed and cylindrically symmetric. We have,

$$\begin{aligned}\boldsymbol{A}(\mathbf{r}) &= \int_V \frac{\mu\boldsymbol{J}(\mathbf{r}')\,e^{-jkR}}{4\pi R}\,d^3\mathbf{r}' = \hat{\mathbf{z}}\,\frac{\mu}{4\pi}\int_V I(z')\,\delta(x')\,\delta(y')\,\frac{e^{-jkR}}{R}\,dx'dy'dz' \\ &= \hat{\mathbf{z}}\,\frac{\mu}{4\pi}\int_L I(z')\,\frac{e^{-jkR}}{R}\,dz'\end{aligned}$$

where $R = |\mathbf{r}-\mathbf{r}'| = \sqrt{\rho^2 + (z-z')^2}$, as shown in Fig. 14.4.1. The z'-integration is over the finite length of the antenna. Thus, $\boldsymbol{A}(\mathbf{r}) = \hat{\mathbf{z}}\,A_z(\rho,z)$, with

$$\boxed{A_z(\rho,z) = \frac{\mu}{4\pi}\int_L I(z')\,\frac{e^{-jkR}}{R}\,dz'}\,, \qquad R = \sqrt{\rho^2 + (z-z')^2} \tag{14.4.3}$$

This is the solution of the z-component of the Helmholtz equation (14.3.2):

$$\nabla^2 A_z + k^2 A_z = -\mu I(z)\,\delta(x)\,\delta(y)$$

Because of the cylindrical symmetry, we can set $\partial/\partial\phi = 0$. Therefore, the gradient and Laplacian operators are $\boldsymbol{\nabla} = \hat{\boldsymbol{\rho}}\,\partial_\rho + \hat{\mathbf{z}}\,\partial_z$ and $\nabla^2 = \rho^{-1}\partial_\rho(\rho\partial_\rho) + \partial_z^2$. Thus, the Helmholtz equation can be written in the form:

$$\frac{1}{\rho}\partial_\rho(\rho\partial_\rho A_z) + \partial_z^2 A_z + k^2 A_z = -\mu I(z)\,\delta(x)\,\delta(y)$$

Away from the antenna, we obtain the homogeneous equation:

$$\frac{1}{\rho}\partial_\rho(\rho\partial_\rho A_z) + \partial_z^2 A_z + k^2 A_z = 0 \tag{14.4.4}$$

Noting that $\boldsymbol{\nabla}\cdot\boldsymbol{A} = \partial_z A_z$, we have from the Lorenz condition:

$$\boxed{\varphi = -\frac{1}{j\omega\mu\epsilon}\partial_z A_z} \qquad \text{(scalar potential of wire antenna)} \tag{14.4.5}$$

The z-component of the electric field is from Eq. (14.3.7):

$$j\omega\mu\epsilon E_z = \partial_z(\nabla\cdot\boldsymbol{A}) + k^2 A_z = \partial_z^2 A_z + k^2 A_z$$

and the radial component:

$$j\omega\mu\epsilon E_\rho = \partial_\rho(\nabla\cdot\boldsymbol{A}) = \partial_\rho\partial_z A_z$$

Using $\boldsymbol{B} = \nabla\times\boldsymbol{A} = (\hat{\boldsymbol{\rho}}\,\partial_\rho + \hat{\mathbf{z}}\,\partial_z)\times(\hat{\mathbf{z}}\,A_z) = (\hat{\boldsymbol{\rho}}\times\hat{\mathbf{z}})\,\partial_\rho A_z = -\hat{\boldsymbol{\phi}}\,\partial_\rho A_z$, it follows that the magnetic field has only a ϕ-component given by $B_\phi = -\partial_\rho A_z$. To summarize, the non-zero field components are all expressible in terms of A_z as follows:

$$\boxed{\begin{aligned} j\omega\mu\epsilon E_z &= \partial_z^2 A_z + k^2 A_z \\ j\omega\mu\epsilon E_\rho &= \partial_\rho\partial_z A_z \\ \mu H_\phi &= -\partial_\rho A_z \end{aligned}} \qquad \text{(fields of a wire antenna)} \tag{14.4.6}$$

Using Eq. (14.4.4), we may re-express E_z in the form:

$$j\omega\mu\epsilon E_z = -\frac{1}{\rho}\partial_\rho(\rho\partial_\rho A_z) = \mu\,\frac{1}{\rho}\partial_\rho(\rho H_\phi) \tag{14.4.7}$$

This is, of course, equivalent to the z-component of Ampère's law. In fact, an even more convenient way to construct the fields is to use the first of Eqs. (14.4.6) to construct E_z and then integrate Eq. (14.4.7) to get H_ϕ and then use the ρ-component of Ampère's law to get E_ρ. The resulting system of equations is:

$$\boxed{\begin{aligned} j\omega\mu\epsilon E_z &= \partial_z^2 A_z + k^2 A_z \\ \partial_\rho(\rho H_\phi) &= j\omega\epsilon\,\rho E_z \\ j\omega\epsilon E_\rho &= -\partial_z H_\phi \end{aligned}} \tag{14.4.8}$$

In Chap. 21, we use (14.4.6) to obtain the Hallén and Pocklington integral equations for determining the current $I(z)$ on a linear antenna, and solve them numerically. In Chap. 22, we use (14.4.8) under the assumption that the current $I(z)$ is sinusoidal to determine the near fields, and use them to compute the self and mutual impedances between linear antennas. The sinusoidal assumption for the current allows us to find E_z, and hence the rest of the fields, *without* having to find A_z first!

14.5 Fields of Electric and Magnetic Dipoles

Finding the fields produced by time-varying electric dipoles has been historically important and has served as a prototypical example for radiation problems.

We consider a point dipole located at the origin, in vacuum, with electric dipole moment $\mathbf{p}$. Assuming harmonic time dependence $e^{j\omega t}$, the corresponding polarization (dipole moment per unit volume) will be: $\boldsymbol{P}(\mathbf{r}) = \mathbf{p}\,\delta^{(3)}(\mathbf{r})$. We saw in Eq. (1.3.18) that the corresponding polarization current and charge densities are:

$$\boldsymbol{J} = \frac{\partial\boldsymbol{P}}{\partial t} = j\omega\boldsymbol{P}, \quad \rho = -\nabla\cdot\boldsymbol{P} \tag{14.5.1}$$

Therefore,

$$\boldsymbol{J}(\mathbf{r}) = j\omega\,\mathbf{p}\,\delta^{(3)}(\mathbf{r})\,, \quad \rho(\mathbf{r}) = -\mathbf{p}\cdot\nabla\delta^{(3)}(\mathbf{r}) \tag{14.5.2}$$

Because of the presence of the delta functions, the integrals in Eq. (14.3.3) can be done trivially, resulting in the vector and scalar potentials:

$$\begin{aligned} \boldsymbol{A}(\mathbf{r}) &= \mu_0\int j\omega\,\mathbf{p}\,\delta^{(3)}(\mathbf{r}')\,G(\mathbf{r}-\mathbf{r}')\,dV' = j\omega\mu_0\,\mathbf{p}\,G(\mathbf{r}) \\ \varphi(\mathbf{r}) &= -\frac{1}{\epsilon_0}\int[\mathbf{p}\cdot\nabla'\delta^{(3)}(\mathbf{r}')]\,G(\mathbf{r}-\mathbf{r}')\,dV' = -\frac{1}{\epsilon_0}\,\mathbf{p}\cdot\nabla G(\mathbf{r}) \end{aligned} \tag{14.5.3}$$

where the integral for φ was done by parts. Alternatively, φ could have been determined from the Lorenz-gauge condition $\nabla\cdot\boldsymbol{A} + j\omega\mu_0\epsilon_0\varphi = 0$.

The $\boldsymbol{E},\boldsymbol{H}$ fields are computed from Eq. (14.3.6), or from (14.3.7), or away from the origin from (14.3.9). We find, where $k^2 = \omega^2/c_0^2 = \omega^2\mu_0\epsilon_0$:

$$\boxed{\begin{aligned} \boldsymbol{E}(\mathbf{r}) &= \frac{1}{\epsilon_0}\,\nabla\times[\nabla G(\mathbf{r})\times\mathbf{p}] = \frac{1}{\epsilon_0}[k^2\,\mathbf{p} + (\mathbf{p}\cdot\nabla)\nabla]\,G(\mathbf{r}) \\ \boldsymbol{H}(\mathbf{r}) &= j\omega\,\nabla G(\mathbf{r})\times\mathbf{p} \end{aligned}} \tag{14.5.4}$$

for $\mathbf{r}\neq 0$. The Green's function $G(\mathbf{r})$ and its gradient are:

$$G(\mathbf{r}) = \frac{e^{-jkr}}{4\pi r}\,, \quad \nabla G(\mathbf{r}) = -\hat{\mathbf{r}}\big(jk + \frac{1}{r}\big)G(\mathbf{r}) = -\hat{\mathbf{r}}\big(jk + \frac{1}{r}\big)\frac{e^{-jkr}}{4\pi r}$$

where $r = |\mathbf{r}|$ and $\hat{\mathbf{r}}$ is the radial unit vector $\hat{\mathbf{r}} = \mathbf{r}/r$. Inserting these into Eq. (14.5.4), we obtain the more explicit expressions:

$$\begin{aligned} \boldsymbol{E}(\mathbf{r}) &= \frac{1}{\epsilon_0}\big(jk + \frac{1}{r}\big)\left[\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{p}) - \mathbf{p}}{r}\right]G(\mathbf{r}) + \frac{k^2}{\epsilon_0}\,\hat{\mathbf{r}}\times(\mathbf{p}\times\hat{\mathbf{r}})\,G(\mathbf{r}) \\ \boldsymbol{H}(\mathbf{r}) &= j\omega\big(jk + \frac{1}{r}\big)(\mathbf{p}\times\hat{\mathbf{r}})\,G(\mathbf{r}) \end{aligned} \tag{14.5.5}$$

If the dipole is moved to location $\mathbf{r}_0$, so that $\boldsymbol{P}(\mathbf{r}) = \mathbf{p}\,\delta^{(3)}(\mathbf{r}-\mathbf{r}_0)$, then the fields are still given by Eqs. (14.5.4) and (14.5.5), with the replacement $G(\mathbf{r})\to G(\mathbf{R})$ and $\hat{\mathbf{r}}\to\hat{\mathbf{R}}$, where $\mathbf{R} = \mathbf{r}-\mathbf{r}_0$.

Eqs. (14.5.5) describe both the near fields and the radiated fields. The limit $\omega = 0$ (or $k = 0$) gives rise to the usual electrostatic dipole electric field, decreasing like $1/r^3$. On the other hand, as we discuss in Sec. 14.7, the radiated fields correspond to the terms decreasing like $1/r$. These are (with $\eta_0 = \sqrt{\mu_0/\epsilon_0}$):

$$\begin{aligned} \boldsymbol{E}_{\text{rad}}(\mathbf{r}) &= \frac{k^2}{\epsilon_0}\,\hat{\mathbf{r}}\times(\mathbf{p}\times\hat{\mathbf{r}})\,G(\mathbf{r}) = \frac{k^2}{\epsilon_0}\,\hat{\mathbf{r}}\times(\mathbf{p}\times\hat{\mathbf{r}})\,\frac{e^{-jkr}}{4\pi r} \\ \boldsymbol{H}_{\text{rad}}(\mathbf{r}) &= j\omega\, jk\,(\mathbf{p}\times\hat{\mathbf{r}})\,G(\mathbf{r}) = \frac{k^2}{\eta_0\epsilon_0}\,(\hat{\mathbf{r}}\times\mathbf{p})\,\frac{e^{-jkr}}{4\pi r} \end{aligned} \tag{14.5.6}$$

They are related by $\eta_0\boldsymbol{H}_{\text{rad}} = \hat{\mathbf{r}}\times\boldsymbol{E}_{\text{rad}}$, which is a general relationship for radiation fields. The same expressions can also be obtained quickly from Eq. (14.5.4) by the substitution rule $\nabla\to -jk\hat{\mathbf{r}}$, discussed in Sec. 14.10.

The near-field, non-radiating, terms in (14.5.5) that drop faster than $1/r$ are important in the new area of *near-field optics* [494–514]. Nanometer-sized dielectric tips (constructed from a tapered fiber) act as tiny dipoles that can probe the evanescent fields from objects, resulting in a dramatic increase (by factors of ten) of the resolution of optical microscopy beyond the Rayleigh diffraction limit and down to atomic scales.

A magnetic dipole at the origin, with magnetic dipole moment $\mathbf{m}$, will be described by the magnetization vector $\boldsymbol{M} = \mathbf{m}\,\delta^{(3)}(\mathbf{r})$. According to Sec. 1.3, the corresponding magnetization current will be $\boldsymbol{J} = \nabla\times\boldsymbol{M} = \nabla\delta^{(3)}(\mathbf{r})\times\mathbf{m}$. Because $\nabla\cdot\boldsymbol{J} = 0$, there is no magnetic charge density, and hence, no scalar potential φ. The vector potential will be:

$$\boldsymbol{A}(\mathbf{r}) = \mu_0\int\nabla\delta^{(3)}(\mathbf{r})\times\mathbf{m}\,G(\mathbf{r}-\mathbf{r}')\,dV' = \mu_0\,\nabla G(\mathbf{r})\times\mathbf{m} \tag{14.5.7}$$

It then follows from Eq. (14.3.6) that:

$$\boxed{\begin{aligned} \boldsymbol{E}(\mathbf{r}) &= -j\omega\mu_0\nabla G(\mathbf{r})\times\mathbf{m} \\ \boldsymbol{H}(\mathbf{r}) &= \nabla\times\big[\nabla G(\mathbf{r})\times\mathbf{m}\big] = \big[k^2\,\mathbf{m} + (\mathbf{m}\cdot\nabla)\nabla\big]G(\mathbf{r}) \end{aligned}} \tag{14.5.8}$$

which become explicitly,

$$\begin{aligned} \boldsymbol{E}(\mathbf{r}) &= j\omega\mu_0\big(jk+\frac{1}{r}\big)\,(\hat{\mathbf{r}}\times\mathbf{m})\,G(\mathbf{r}) \\ \boldsymbol{H}(\mathbf{r}) &= \big(jk+\frac{1}{r}\big)\left[\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{m})-\mathbf{m}}{r}\right]G(\mathbf{r}) + k^2\,\hat{\mathbf{r}}\times(\mathbf{m}\times\hat{\mathbf{r}})\,G(\mathbf{r}) \end{aligned} \tag{14.5.9}$$

The corresponding radiation fields are:

$$\begin{aligned} \boldsymbol{E}_{\text{rad}}(\mathbf{r}) &= j\omega\mu_0\, jk\,(\hat{\mathbf{r}}\times\mathbf{m})\,G(\mathbf{r}) = \eta_0 k^2\,(\mathbf{m}\times\hat{\mathbf{r}})\,\frac{e^{-jkr}}{4\pi r} \\ \boldsymbol{H}_{\text{rad}}(\mathbf{r}) &= k^2\,\hat{\mathbf{r}}\times(\mathbf{m}\times\hat{\mathbf{r}})\,G(\mathbf{r}) = k^2\,\hat{\mathbf{r}}\times(\mathbf{m}\times\hat{\mathbf{r}})\,\frac{e^{-jkr}}{4\pi r} \end{aligned} \tag{14.5.10}$$

We note that the fields of the magnetic dipole are obtained from those of the electric dipole by the *duality* transformations $\boldsymbol{E}\to\boldsymbol{H}$, $\boldsymbol{H}\to-\boldsymbol{E}$, $\epsilon_0\to\mu_0$, $\mu_0\to\epsilon_0$, $\eta_0\to 1/\eta_0$, and $\mathbf{p}\to\mu_0\,\mathbf{m}$, that latter following by comparing the terms $\boldsymbol{P}$ and $\mu_0\boldsymbol{M}$ in the constitutive relations (1.3.16). Duality is discussed in more detail in Sec. 17.2.

The electric and magnetic dipoles are essentially equivalent to the linear and loop Hertzian dipole antennas, respectively, which are discussed in sections 16.2 and 16.8. Problem 14.4 establishes the usual results $\mathbf{p} = Q\,\mathbf{d}$ for a pair of charges $\pm Q$ separated by a distance $\mathbf{d}$, and $\mathbf{m} = \hat{\mathbf{z}}\,IS$ for a current loop of area S.

Example 14.5.1: We derive explicit expressions for the real-valued electric and magnetic fields of an oscillating z-directed dipole $\mathbf{p}(t) = p\,\hat{\mathbf{z}}\cos\omega t$. And also derive and plot the electric field lines at several time instants. This problem has an important history, having been considered first by Hertz in 1889 in a paper reprinted in [56].

Restoring the $e^{j\omega t}$ factor in Eq. (14.5.5) and taking real parts, we obtain the fields:

$$\begin{aligned} \boldsymbol{\mathcal{E}}(\mathbf{r}) &= p\big[k\sin(kr-\omega t) + \frac{\cos(kr-\omega t)}{r}\big]\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\hat{\mathbf{z}})-\hat{\mathbf{z}}}{4\pi\epsilon_0 r^2} + \frac{pk^2\,\hat{\mathbf{r}}\times(\hat{\mathbf{z}}\times\hat{\mathbf{r}})}{4\pi\epsilon_0 r}\cos(kr-\omega t) \\ \boldsymbol{\mathcal{H}}(\mathbf{r}) &= p\omega\big[-k\cos(kr-\omega t) + \frac{\sin(kr-\omega t)}{r}\big]\left[\frac{\hat{\mathbf{z}}\times\hat{\mathbf{r}}}{4\pi r}\right] \end{aligned}$$

In spherical coordinates, we have $\hat{\mathbf{z}} = \hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$. This gives $3\,\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\hat{\mathbf{z}})-\hat{\mathbf{z}} = 2\,\hat{\mathbf{r}}\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta$, $\hat{\mathbf{r}}\times(\hat{\mathbf{z}}\times\hat{\mathbf{r}}) = -\hat{\boldsymbol{\theta}}\sin\theta$, and $\hat{\mathbf{z}}\times\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}\sin\theta$. Therefore, the non-zero components of $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{H}}$ are $\mathcal{E}_r, \mathcal{E}_\phi$ and $\mathcal{H}_\phi$:

$$\begin{aligned} \mathcal{E}_r(\mathbf{r}) &= p\big[k\sin(kr-\omega t) + \frac{\cos(kr-\omega t)}{r}\big]\left[\frac{2\cos\theta}{4\pi\epsilon_0 r^2}\right] \\ \mathcal{E}_\theta(\mathbf{r}) &= p\big[k\sin(kr-\omega t) + \frac{\cos(kr-\omega t)}{r}\big]\left[\frac{\sin\theta}{4\pi\epsilon_0 r^2}\right] - \frac{pk^2\,\sin\theta}{4\pi\epsilon_0 r}\cos(kr-\omega t) \\ \mathcal{H}_\phi(\mathbf{r}) &= p\omega\big[-k\cos(kr-\omega t) + \frac{\sin(kr-\omega t)}{r}\big]\left[\frac{\sin\theta}{4\pi r}\right] \end{aligned}$$

By definition, the electric field is tangential to its field lines. A small displacement $d\mathbf{r}$ along the tangent to a line will be parallel to $\boldsymbol{\mathcal{E}}$ at that point. This implies that $d\mathbf{r}\times\boldsymbol{\mathcal{E}} = 0$, which can be used to determine the lines. Because of the azimuthal symmetry in the ϕ variable, we may look at the field lines that lie on the xz-plane (that is, $\phi = 0$). Then, we have:

$$d\mathbf{r}\times\boldsymbol{\mathcal{E}} = (\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\theta}}\,r\,d\theta)\times(\hat{\mathbf{r}}\,\mathcal{E}_r + \hat{\boldsymbol{\theta}}\,\mathcal{E}_\theta) = \hat{\boldsymbol{\phi}}\,(dr\,\mathcal{E}_\theta - r\,d\theta\,\mathcal{E}_r) = 0 \quad\Rightarrow\quad \frac{dr}{d\theta} = \frac{r\,\mathcal{E}_r}{\mathcal{E}_\theta}$$

This determines r as a function of θ, giving the polar representation of the line curve. To solve this equation, we rewrite the electric field in terms of the dimensionless variables $u = kr$ and $\delta = \omega t$, defining $E_0 = pk^3/4\pi\epsilon_0$:

$$\begin{aligned} \mathcal{E}_r &= E_0\,\frac{2\cos\theta}{u^2}\left[\sin(u-\delta) + \frac{\cos(u-\delta)}{u}\right] \\ \mathcal{E}_\theta &= -E_0\,\frac{\sin\theta}{u}\left[\cos(u-\delta) - \frac{\cos(u-\delta)}{u^2} - \frac{\sin(u-\delta)}{u}\right] \end{aligned}$$

We note that the factors within the square brackets are related by differentiation:

$$\begin{aligned} Q(u) &= \sin(u-\delta) + \frac{\cos(u-\delta)}{u} \\ Q'(u) &= \frac{dQ(u)}{du} = \cos(u-\delta) - \frac{\cos(u-\delta)}{u^2} - \frac{\sin(u-\delta)}{u} \end{aligned}$$

Therefore, the fields are:

$$\mathcal{E}_r = E_0\,\frac{2\cos\theta}{u^2}\,Q(u)\,,\quad \mathcal{E}_\theta = -E_0\,\frac{\sin\theta}{u}\,Q'(u)$$

It follows that the equation for the lines in the variable u will be:

$$\frac{du}{d\theta} = \frac{u\mathcal{E}_r}{\mathcal{E}_\theta} = -2\cot\theta\left[\frac{Q(u)}{Q'(u)}\right] \quad\Rightarrow\quad \frac{d}{d\theta}\big[\ln Q(u)\big] = -2\cot\theta = -\frac{d}{d\theta}\big[\ln\sin^2\theta\big]$$

which gives:

$$\frac{d}{d\theta}\ln\big[Q(u)\sin^2\theta\big] = 0 \quad\Rightarrow\quad Q(u)\sin^2\theta = C$$

where C is a constant. Thus, the electric field lines are given implicitly by:

$$\left[\sin(u-\delta)+\frac{\cos(u-\delta)}{u}\right]\sin^2\theta = \left[\sin(kr-\omega t)+\frac{\cos(kr-\omega t)}{kr}\right]\sin^2\theta = C$$

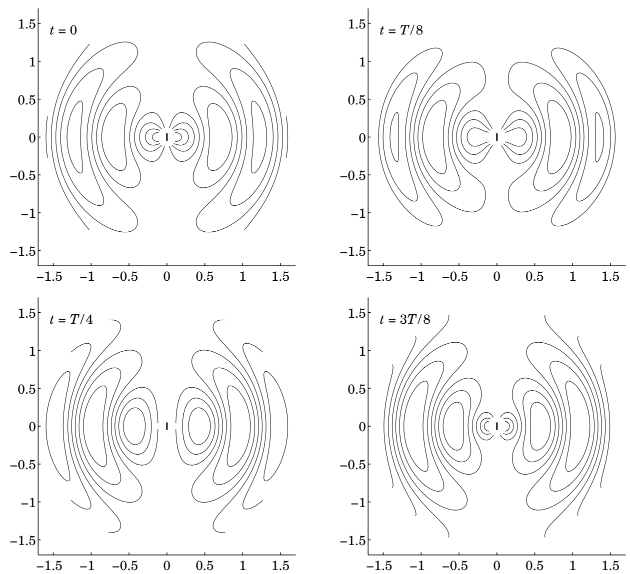


Fig. 14.5.1 Electric field lines of oscillating dipole at successive time instants.

Ideally, one should solve for r in terms of θ. Because this is not possible in closed form, we prefer to think of the lines as a contour plot at different values of the constant C. The resulting graphs are shown in Fig. 14.5.1. They were generated at the four time instants $t = 0$, $T/8$, $T/4$, and $3T/8$, where T is the period of oscillation, $T = 2\pi/\omega$. The x, z distances are in units of λ and extend to 1.5λ. The dipole is depicted as a tiny z-directed line at the origin. The following MATLAB code illustrates the generation of these plots:

```
rmin = 1/8; rmax = 1.6;                  % plot limits in wavelengths λ
Nr = 61; Nth = 61; N = 6;                % meshpoints and number of contour levels
t = 1/8; d = 2*pi*t;                     % time instant t = T/8

[r,th] = meshgrid(linspace(rmin,rmax,Nr), linspace(0,pi,Nth));
```



```
u = 2*pi*r;                              % r is in units of λ
z = r.*cos(th); x = r.*sin(th);          % cartesian coordinates in units of λ

C = (cos(u-d)./u + sin(u-d)) .* sin(th).^2;        % contour levels

contour([-x; x], [z; z], [C; C], N);     % right and left-reflected contours with N levels
```

We observe how the lines form closed loops originating at the dipole. The loops eventually escape the vicinity of the dipole and move outwards, pushing away the loops that are ahead of them. In this fashion, the field gets radiated away from its source. The MATLAB file `dipmovie.m` generates a movie of the evolving field lines lasting from $t = 0$ to $t = 8T$. □

14.6 Ewald-Oseen Extinction Theorem

The reflected and transmitted fields of a plane wave incident on a dielectric were determined in Chapters 5 and 7 by solving the wave equations in each medium and matching the solutions at the interface by imposing the boundary conditions.

Although this approach yields the correct solutions, it hides the physics. From the microscopic point of view, the dielectric consists of polarizable atoms or molecules, each of which is radiating *in vacuum* in response to the incident field and in response to the fields radiated by the other atoms. The total radiated field must combine with the incident field so as to generate the correct transmitted field. This is the essence of the Ewald-Oseen extinction theorem [458–493]. The word "extinction" refers to the cancellation of the incident field inside the dielectric.

Let $\boldsymbol{E}(\mathbf{r})$ be the incident field, $\boldsymbol{E}_{\text{rad}}(\mathbf{r})$ the total radiated field, and $\boldsymbol{E}'(\mathbf{r})$ the transmitted field in the dielectric. Then, the theorem states that (for $\mathbf{r}$ inside the dielectric):

$$\boldsymbol{E}_{\text{rad}}(\mathbf{r}) = \boldsymbol{E}'(\mathbf{r}) - \boldsymbol{E}(\mathbf{r}) \quad\Rightarrow\quad \boxed{\boldsymbol{E}'(\mathbf{r}) = \boldsymbol{E}(\mathbf{r}) + \boldsymbol{E}_{\text{rad}}(\mathbf{r})} \tag{14.6.1}$$

We will follow a simplified approach to the extinction theorem as in Refs. [479–493] and in particular [493]. We assume that the incident field is a uniform plane wave, with TE or TM polarization, incident obliquely on a planar dielectric interface, as shown in Fig. 14.6.1. The incident and transmitted fields will have the form:

$$\boldsymbol{E}(\mathbf{r}) = \boldsymbol{E}_0\, e^{-j\,\boldsymbol{k}\cdot\mathbf{r}}, \quad \boldsymbol{E}'(\mathbf{r}) = \boldsymbol{E}'_0\, e^{-j\,\boldsymbol{k}'\cdot\mathbf{r}} \tag{14.6.2}$$

The expected relationships between the transmitted and incident waves were summarized in Eqs. (7.7.1)–(7.7.5). We will derive the same results from the present approach. The incident wave vector is $\boldsymbol{k} = k_x\hat{\mathbf{z}} + k_z\hat{\mathbf{z}}$ with $k = \omega/c_0 = \omega\sqrt{\epsilon_0\mu_0}$, and satisfies $\boldsymbol{k}\cdot\boldsymbol{E}_0 = 0$. For the transmitted wave, we will find that $\boldsymbol{k}' = k_x\hat{\mathbf{z}} + k'_z\hat{\mathbf{z}}$ satisfies $\boldsymbol{k}'\cdot\boldsymbol{E}'_0 = 0$ and $k' = \omega/c = \omega\sqrt{\epsilon\mu_0} = kn$, so that $c = c_0/n$, where n is the refractive index of the dielectric, $n = \sqrt{\epsilon/\epsilon_0}$.

The radiated field is given by Eq. (14.3.10), where $\boldsymbol{J}$ is the current due to the polarization $\boldsymbol{P}$, that is, $\boldsymbol{J} = \dot{\boldsymbol{P}} = j\omega\boldsymbol{P}$. Although there is no volume polarization charge density,† there may be a surface polarization density $\rho_s = \hat{\mathbf{n}}\cdot\boldsymbol{P}$ on the planar dielectric interface. Because $\hat{\mathbf{n}} = -\hat{\mathbf{z}}$, we will have $\rho_s = -\hat{\mathbf{z}}\cdot\boldsymbol{P} = -P_z$. Such density is present only in the TM

† $\rho = -\nabla\cdot\boldsymbol{P}$ vanishes for the type of plane-wave solutions that we consider here.

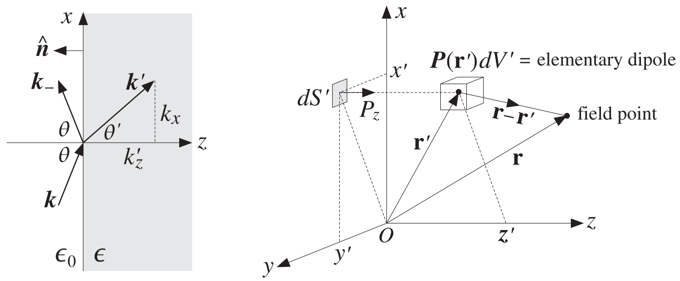


Fig. 14.6.1 Elementary dipole at $\mathbf{r}'$ contributes to the local field at $\mathbf{r}$.

case [493]. The corresponding volume term in Eq. (14.3.10) will collapse into a surface integral. Thus, the field generated by the densities $\boldsymbol{J}, \rho_s$ will be:

$$\boldsymbol{E}_{\rm rad}(\mathbf{r}) = -j\omega\mu_0\int_V \boldsymbol{J}(\mathbf{r}')\,G(\mathbf{r}-\mathbf{r}')\,dV' + \frac{1}{\epsilon_0}\int_S \rho_s(\mathbf{r}')\,\nabla' G(\mathbf{r}-\mathbf{r}')\,dS'$$

where $G(\mathbf{r}) = e^{-jkr}/4\pi r$ is the vacuum Green's function having $k = \omega/c_0$, and V is the right half-space $z \geq 0$, and S, the xy-plane. Replacing $\boldsymbol{J}, \rho_s$ in terms of the polarization and writing $\nabla' G = -\nabla G$, and moving ∇ outside the surface integral, we have:

$$\boldsymbol{E}_{\rm rad}(\mathbf{r}) = \omega^2\mu_0\int_V \boldsymbol{P}(\mathbf{r}')\,G(\mathbf{r}-\mathbf{r}')\,dV' + \frac{1}{\epsilon_0}\nabla\int_S P_z(\mathbf{r}')\,G(\mathbf{r}-\mathbf{r}')\,dS' \tag{14.6.3}$$

We assume that the polarization $\boldsymbol{P}(\mathbf{r}')$ is induced by the total field inside the dielectric, that is, we set $\boldsymbol{P}(\mathbf{r}') = \epsilon_0\chi\,\boldsymbol{E}'(\mathbf{r}')$, where χ is the electric susceptibility. Setting $k^2 = \omega^2\mu_0\epsilon_0$, Eq. (14.6.3) becomes:

$$\boldsymbol{E}_{\rm rad}(\mathbf{r}) = k^2\chi\int_V \boldsymbol{E}'(\mathbf{r}')\,G(\mathbf{r}-\mathbf{r}')\,dV' + \chi\nabla\int_S E'_z(\mathbf{r}')\,G(\mathbf{r}-\mathbf{r}')\,dS' \tag{14.6.4}$$

Evaluated at points $\mathbf{r}$ on the left of the interface ($z < 0$), $\boldsymbol{E}_{\rm rad}(\mathbf{r})$ should generate the *reflected field*. Evaluated within the dielectric ($z \geq 0$), it should give Eq. (14.6.1), resulting in the self-consistency condition:

$$k^2\chi\int_V \boldsymbol{E}'(\mathbf{r}')\,G(\mathbf{r}-\mathbf{r}')\,dV' + \chi\nabla\int_S E'_z(\mathbf{r}')\,G(\mathbf{r}-\mathbf{r}')\,dS' = \boldsymbol{E}'(\mathbf{r}) - \boldsymbol{E}(\mathbf{r}) \tag{14.6.5}$$

Inserting Eq. (14.6.2), we obtain the condition:

$$k^2\chi\,\boldsymbol{E}'_0\int_V e^{-j\boldsymbol{k}'\cdot\mathbf{r}'}G(\mathbf{r}-\mathbf{r}')\,dV' + \chi E'_{z0}\nabla\int_S e^{-j\boldsymbol{k}'\cdot\mathbf{r}'}G(\mathbf{r}-\mathbf{r}')\,dS' = \boldsymbol{E}'_0\,e^{-j\boldsymbol{k}'\cdot\mathbf{r}} - \boldsymbol{E}_0\,e^{-j\boldsymbol{k}\cdot\mathbf{r}}$$

The vector $\boldsymbol{k}' = k'_x\hat{\mathbf{x}} + k'_z\hat{\mathbf{z}}$ may be assumed to have $k'_x = k_x$, which is equivalent to Snel's law. This follows easily from the phase matching of the $e^{jk_x x}$ factors in the above equation. Then, the integrals over S and V can be done easily using Eqs. (D.14) and (D.16) of Appendix D, with (D.14) being evaluated at $z' = 0$ and $z \geq 0$:

$$\begin{aligned}
&\int_V e^{-j\boldsymbol{k}'\cdot\mathbf{r}'}G(\mathbf{r}-\mathbf{r}')\,dV' = \frac{e^{-j\boldsymbol{k}'\cdot\mathbf{r}}}{k'^2-k^2} - \frac{e^{-j\boldsymbol{k}\cdot\mathbf{r}}}{2k_z(k'_z-k_z)}\\
&\int_S e^{-j\boldsymbol{k}'\cdot\mathbf{r}'}G(\mathbf{r}-\mathbf{r}')\,dS' = \frac{e^{-j\boldsymbol{k}\cdot\mathbf{r}}}{2jk_z} \;\Rightarrow\; \nabla\int_S e^{-j\boldsymbol{k}'\cdot\mathbf{r}'}G(\mathbf{r}-\mathbf{r}')\,dS' = -\frac{\boldsymbol{k}\,e^{-j\boldsymbol{k}\cdot\mathbf{r}}}{2k_z}
\end{aligned} \tag{14.6.6}$$

The self-consistency condition reads now:

$$k^2\chi\,\boldsymbol{E}'_0\left[\frac{e^{-j\boldsymbol{k}'\cdot\mathbf{r}}}{k'^2-k^2} - \frac{e^{-j\boldsymbol{k}\cdot\mathbf{r}}}{2k_z(k'_z-k_z)}\right] - \chi E'_{z0}\,\frac{\boldsymbol{k}\,e^{-j\boldsymbol{k}\cdot\mathbf{r}}}{2k_z} = \boldsymbol{E}'_0\,e^{-j\boldsymbol{k}'\cdot\mathbf{r}} - \boldsymbol{E}_0\,e^{-j\boldsymbol{k}\cdot\mathbf{r}}$$

Equating the coefficients of like exponentials, we obtain the two conditions:

$$\frac{k^2\chi}{k'^2-k^2}\,\boldsymbol{E}'_0 = \boldsymbol{E}'_0 \quad\Rightarrow\quad \frac{k^2\chi}{k'^2-k^2} = 1 \quad\Rightarrow\quad k'^2 = k^2(1+\chi) = k^2n^2 \tag{14.6.7}$$

$$\frac{k^2\chi}{2k_z(k'_z-k_z)}\,\boldsymbol{E}'_0 + \frac{\chi\,\boldsymbol{k}}{2k_z}\,E'_{z0} = \boldsymbol{E}_0 \tag{14.6.8}$$

The first condition implies that $k' = kn$, where $n = \sqrt{1+\chi} = \sqrt{\epsilon/\epsilon_0}$. Thus, the phase velocity within the dielectric is $c = c_0/n$. Replacing $\chi = (k'^2-k^2)/k^2 = (k'^2_z-k^2_z)/k^2$, we may rewrite Eq. (14.6.8) as:

$$\frac{k'^2_z-k^2_z}{2k_z(k'_z-k_z)}\,\boldsymbol{E}'_0 + \frac{(k'^2_z-k^2_z)\,\boldsymbol{k}}{2k_z\,k^2}\,E'_{z0} = \boldsymbol{E}_0\,,\quad\text{or,}$$

$$\boldsymbol{E}'_0 + \frac{\boldsymbol{k}}{k^2}(k'_z-k_z)E'_{z0} = \frac{2k_z}{k'_z+k_z}\,\boldsymbol{E}_0 \tag{14.6.9}$$

This implies immediately the transversality condition for the transmitted field, that is, $\boldsymbol{k}'\cdot\boldsymbol{E}'_0 = 0$. Indeed, using $\boldsymbol{k}\cdot\boldsymbol{E}_0 = 0$ for the incident field, we find:

$$\boldsymbol{k}\cdot\boldsymbol{E}'_0 + \frac{\boldsymbol{k}\cdot\boldsymbol{k}}{k^2}(k'_z-k_z)E'_{z0} = \frac{2k_z}{k'_z+k_z}\,\boldsymbol{k}\cdot\boldsymbol{E}_0 = 0 \quad\Rightarrow\quad \boldsymbol{k}\cdot\boldsymbol{E}'_0 + (k'_z-k_z)E'_{z0} = 0$$

or, explicitly, $k_xE'_{x0} + k_zE'_{z0} + (k'_z-k_z)E'_{z0} = k_xE'_{x0} + k'_zE'_{z0} = \boldsymbol{k}'\cdot\boldsymbol{E}'_0 = 0$. Replacing $(k'_z-k_z)E'_{z0} = -\boldsymbol{k}\cdot\boldsymbol{E}'_0$ in Eq. (14.6.9) and using the BAC-CAB rule, we obtain:

$$\boldsymbol{E}'_0 - \frac{\boldsymbol{k}}{k^2}(\boldsymbol{k}\cdot\boldsymbol{E}'_0) = \frac{2k_z}{k'_z+k_z}\,\boldsymbol{E}_0 \quad\Rightarrow\quad \boxed{\frac{\boldsymbol{k}\times(\boldsymbol{E}'_0\times\boldsymbol{k})}{k^2} = \frac{2k_z}{k'_z+k_z}\,\boldsymbol{E}_0} \tag{14.6.10}$$

It can be shown that Eq. (14.6.10) is *equivalent* to the transmission coefficient results summarized in Eqs. (7.7.1)–(7.7.5), for both the TE and TM cases (see also Problem 7.6 and the identities in Problem 7.5.) The transmitted magnetic field $\boldsymbol{H}'(\mathbf{r}) = \boldsymbol{H}'_0\,e^{-j\boldsymbol{k}'\cdot\mathbf{r}}$ may be found from Faraday's law $\nabla\times\boldsymbol{E}' = -j\omega\mu_0\boldsymbol{H}'$, which reads $\omega\mu_0\boldsymbol{H}'_0 = \boldsymbol{k}'\times\boldsymbol{E}'_0$.

Next, we look at the reflected field. For points $\mathbf{r}$ lying to the left of the interface ($z \leq 0$), the evaluation of the integrals (14.6.6) gives according to Eqs. (D.14) and (D.16), where (D.14) is evaluated at $z' = 0$ and $z \leq 0$:

$$\begin{aligned}
&\int_V e^{-j\boldsymbol{k}'\cdot\mathbf{r}'}G(\mathbf{r}-\mathbf{r}')\,dV' = -\frac{e^{-j\boldsymbol{k}_-\cdot\mathbf{r}}}{2k_z(k'_z+k_z)}\\
&\int_S e^{-j\boldsymbol{k}'\cdot\mathbf{r}'}G(\mathbf{r}-\mathbf{r}')\,dS' = \frac{e^{-j\boldsymbol{k}_-\cdot\mathbf{r}}}{2jk_z} \;\Rightarrow\; \nabla\int_S e^{-j\boldsymbol{k}'\cdot\mathbf{r}'}G(\mathbf{r}-\mathbf{r}')\,dS' = -\frac{\boldsymbol{k}_-\,e^{-j\boldsymbol{k}_-\cdot\mathbf{r}}}{2k_z}
\end{aligned}$$

where $\boldsymbol{k}_-$ denotes the reflected wave vector, $\boldsymbol{k}_- = k_x\hat{\mathbf{x}} - k_z\hat{\mathbf{z}}$. It follows that the total radiated field will be:

$$\boldsymbol{E}_{\rm rad}(\mathbf{r}) = k^2\chi\,\boldsymbol{E}'_0\left[-\frac{e^{-j\boldsymbol{k}_-\cdot\mathbf{r}}}{2k_z(k'_z+k_z)}\right] - \frac{\boldsymbol{k}_-\chi E'_{z0}}{2k_z}\,e^{-j\boldsymbol{k}_-\cdot\mathbf{r}} = \boldsymbol{E}_{-0}e^{-j\boldsymbol{k}_-\cdot\mathbf{r}}$$

where the overall coefficient $\boldsymbol{E}_{-0}$ can be written in the form:

$$\boldsymbol{E}_{-0} = -\frac{k^2\chi}{2k_z(k_z'+k_z)}\boldsymbol{E}_0' - \frac{\boldsymbol{k}_-\chi E_{z0}'}{2k_z} = \frac{k_z-k_z'}{2k_z}\left[\boldsymbol{E}_0' + \frac{\boldsymbol{k}_-(k_z'+k_z)E_{z0}'}{k^2}\right]$$

where we set $\chi = (k_z'^2 - k_z^2)/k^2$. Noting the identity $\boldsymbol{k}_-\cdot\boldsymbol{E}_0' + (k_z'+k_z)E_{z0}' = \boldsymbol{k}'\cdot\boldsymbol{E}_0' = 0$ and $\boldsymbol{k}_-\cdot\boldsymbol{k}_- = k^2$, we finally find:

$$\boldsymbol{E}_{-0} = \frac{k_z-k_z'}{2k_z}\left[\boldsymbol{E}_0' - \frac{\boldsymbol{k}_-(\boldsymbol{k}_-\cdot\boldsymbol{E}_0')}{k^2}\right] \quad\Rightarrow\quad \boxed{\frac{\boldsymbol{k}_-\times(\boldsymbol{E}_0'\times\boldsymbol{k}_-)}{k^2} = \frac{2k_z}{k_z-k_z'}\boldsymbol{E}_{-0}} \tag{14.6.11}$$

It can be verified that (14.6.11) is equivalent to the reflected fields as given by Eqs. (7.7.1)–(7.7.5) for the TE and TM cases. We note also that $\boldsymbol{k}_-\cdot\boldsymbol{E}_{-0} = 0$.

The conventional boundary conditions are a *consequence* of this approach. For example, Eqs. (14.6.10) and (14.6.11) imply the continuity of the tangential components of the $\boldsymbol{E}$-field. Indeed, we find by adding:

$$\boldsymbol{E}_0 + \boldsymbol{E}_{-0} = \boldsymbol{E}_0' + \frac{\chi E_{z0}'}{2k_z}(\boldsymbol{k}-\boldsymbol{k}_-) = \boldsymbol{E}_0' + \chi\,\hat{\mathbf{z}}\,E_{z0}'$$

which implies that $\hat{\mathbf{z}}\times(\boldsymbol{E}_0+\boldsymbol{E}_{-0}) = \hat{\mathbf{z}}\times\boldsymbol{E}_0'$.

In summary, the radiated fields from the polarizable atoms cause the cancellation of the incident vacuum field throughout the dielectric and conspire to generate the correct transmitted field that has phase velocity $c = c_0/n$. The reflected wave does not originate just at the interface but rather it is the field radiated backwards by the atoms within the entire body of the dielectric.

Next, we discuss another simplified approach based on radiating dipoles [484]. It has the additional advantage that it leads to the Lorentz-Lorenz or Clausius-Mossotti relationship between refractive index and polarizability. General proofs of the extinction theorem may be found in [458–478] and [598].

The dielectric is viewed as a collection of dipoles $\mathbf{p}_i$ at locations $\mathbf{r}_i$. The dipole moments are assumed to be induced by a local (or effective) electric field $\boldsymbol{E}_{\text{loc}}(\mathbf{r})$ through $\mathbf{p}_i = \alpha\epsilon_0\boldsymbol{E}_{\text{loc}}(\mathbf{r}_i)$, where α is the polarizability.† The field radiated by the jth dipole $\mathbf{p}_j$ is given by Eq. (14.5.4), where $G(\mathbf{r})$ is the vacuum Green's function:

$$\boldsymbol{E}_j(\mathbf{r}) = \frac{1}{\epsilon_0}\nabla\times\nabla\times\left[\mathbf{p}_j\,G(\mathbf{r}-\mathbf{r}_j)\right]$$

The field at the location of the ith dipole due to all the other dipoles will be:

$$\boldsymbol{E}_{\text{rad}}(\mathbf{r}_i) = \sum_{j\neq i}\boldsymbol{E}_j(\mathbf{r}_i) = \frac{1}{\epsilon_0}\sum_{j\neq i}\nabla_i\times\nabla_i\times\left[\mathbf{p}_j\,G(\mathbf{r}_i-\mathbf{r}_j)\right] \tag{14.6.12}$$

where ∇_i is with respect to $\mathbf{r}_i$. Passing to a continuous description, we assume N dipoles per unit volume, so that the polarization density will be $\boldsymbol{P}(\mathbf{r}') = N\,\mathbf{p}(\mathbf{r}') = N\alpha\epsilon_0\boldsymbol{E}_{\text{loc}}(\mathbf{r}')$. Then, Eq. (14.6.12) is replaced by the (principal-value) integral:

$$\boldsymbol{E}_{\text{rad}}(\mathbf{r}) = \frac{1}{\epsilon_0}\int_V\Big[\nabla\times\nabla\times\left[\boldsymbol{P}(\mathbf{r}')\,G(\mathbf{r}-\mathbf{r}')\right]\Big]_{\mathbf{r}'\neq\mathbf{r}}dV' \tag{14.6.13}$$

†Normally, the polarizability is defined as the quantity $\alpha' = \alpha\epsilon_0$.

Using Eq. (D.7) of Appendix D, we rewrite:

$$\boldsymbol{E}_{\text{rad}}(\mathbf{r}) = \frac{1}{\epsilon_0}\nabla\times\nabla\times\int_V\boldsymbol{P}(\mathbf{r}')\,G(\mathbf{r}-\mathbf{r}')\,dV' - \frac{2}{3\epsilon_0}\boldsymbol{P}(\mathbf{r}) \tag{14.6.14}$$

and in terms of the local field ($N\alpha$ is dimensionless):

$$\boldsymbol{E}_{\text{rad}}(\mathbf{r}) = N\alpha\,\nabla\times\nabla\times\int_V\boldsymbol{E}_{\text{loc}}(\mathbf{r}')\,G(\mathbf{r}-\mathbf{r}')\,dV' - \frac{2}{3}N\alpha\,\boldsymbol{E}_{\text{loc}}(\mathbf{r}) \tag{14.6.15}$$

According to the Ewald-Oseen extinction requirement, the radiated field must cancel the incident field $\boldsymbol{E}(\mathbf{r})$ while generating the local field $\boldsymbol{E}_{\text{loc}}(\mathbf{r})$, that is, $\boldsymbol{E}_{\text{rad}}(\mathbf{r}) = \boldsymbol{E}_{\text{loc}}(\mathbf{r}) - \boldsymbol{E}(\mathbf{r})$. This leads to the self-consistency condition:

$$N\alpha\,\nabla\times\nabla\times\int_V\boldsymbol{E}_{\text{loc}}(\mathbf{r}')\,G(\mathbf{r}-\mathbf{r}')\,dV' - \frac{2}{3}N\alpha\,\boldsymbol{E}_{\text{loc}}(\mathbf{r}) = \boldsymbol{E}_{\text{loc}}(\mathbf{r}) - \boldsymbol{E}(\mathbf{r}) \tag{14.6.16}$$

Assuming a plane-wave solution $\boldsymbol{E}_{\text{loc}}(\mathbf{r}) = \boldsymbol{E}_1'\,e^{-j\boldsymbol{k}'\cdot\mathbf{r}}$, we obtain:

$$N\alpha\,\nabla\times\nabla\times\boldsymbol{E}_1'\int_V e^{-j\boldsymbol{k}'\cdot\mathbf{r}}\,G(\mathbf{r}-\mathbf{r}')\,dV' - \frac{2}{3}N\alpha\,\boldsymbol{E}_1'\,e^{-j\boldsymbol{k}'\cdot\mathbf{r}} = \boldsymbol{E}_1'\,e^{-j\boldsymbol{k}'\cdot\mathbf{r}} - \boldsymbol{E}_0\,e^{-j\boldsymbol{k}\cdot\mathbf{r}}$$

For $\mathbf{r}$ within the dielectric, we find as before:

$$N\alpha\,\nabla\times\nabla\times\boldsymbol{E}_1'\left[\frac{e^{-j\boldsymbol{k}'\cdot\mathbf{r}}}{k'^2-k^2} - \frac{e^{-j\boldsymbol{k}\cdot\mathbf{r}}}{2k_z(k_z'-k_z)}\right] - \frac{2}{3}N\alpha\,\boldsymbol{E}_1'\,e^{-j\boldsymbol{k}'\cdot\mathbf{r}} = \boldsymbol{E}_1'\,e^{-j\boldsymbol{k}'\cdot\mathbf{r}} - \boldsymbol{E}_0\,e^{-j\boldsymbol{k}\cdot\mathbf{r}}$$

$$N\alpha\,\nabla\times\nabla\times\boldsymbol{E}_1'\left[\frac{e^{-j\boldsymbol{k}'\cdot\mathbf{r}}}{k'^2-k^2} - \frac{e^{-j\boldsymbol{k}\cdot\mathbf{r}}}{2k_z(k_z'-k_z)}\right] = \left(1+\frac{2}{3}N\alpha\right)\boldsymbol{E}_1'\,e^{-j\boldsymbol{k}'\cdot\mathbf{r}} - \boldsymbol{E}_0\,e^{-j\boldsymbol{k}\cdot\mathbf{r}}$$

Performing the ∇ operations, we have:

$$N\alpha\left[\frac{\boldsymbol{k}'\times(\boldsymbol{E}_1'\times\boldsymbol{k}')}{k'^2-k^2}e^{-j\boldsymbol{k}'\cdot\mathbf{r}} - \frac{\boldsymbol{k}\times(\boldsymbol{E}_1'\times\boldsymbol{k})}{2k_z(k_z'-k_z)}e^{-j\boldsymbol{k}\cdot\mathbf{r}}\right] = \left(1+\frac{2}{3}N\alpha\right)\boldsymbol{E}_1'\,e^{-j\boldsymbol{k}'\cdot\mathbf{r}} - \boldsymbol{E}_0\,e^{-j\boldsymbol{k}\cdot\mathbf{r}}$$

Equating the coefficients of the exponentials, we obtain the two conditions:

$$N\alpha\,\frac{\boldsymbol{k}'\times(\boldsymbol{E}_1'\times\boldsymbol{k}')}{k'^2-k^2} = \left(1+\frac{2}{3}N\alpha\right)\boldsymbol{E}_1' \tag{14.6.17}$$

$$N\alpha\,\frac{\boldsymbol{k}\times(\boldsymbol{E}_1'\times\boldsymbol{k})}{2k_z(k_z'-k_z)} = \boldsymbol{E}_0 \tag{14.6.18}$$

The first condition implies immediately that $\boldsymbol{k}'\cdot\boldsymbol{E}_1' = 0$, therefore, using the BAC-CAB rule, the condition reads:

$$\frac{N\alpha\,k'^2}{k'^2-k^2}\boldsymbol{E}_1' = \left(1+\frac{2}{3}N\alpha\right)\boldsymbol{E}_1' \quad\Rightarrow\quad \frac{N\alpha\,k'^2}{k'^2-k^2} = 1+\frac{2}{3}N\alpha \tag{14.6.19}$$

Setting $k' = kn$, Eq. (14.6.19) implies the *Lorentz-Lorenz formula*:

$$\frac{N\alpha\,n^2}{n^2-1} = 1+\frac{2}{3}N\alpha \quad\Rightarrow\quad \boxed{\frac{n^2-1}{n^2+2} = \frac{1}{3}N\alpha} \tag{14.6.20}$$

We must distinguish between the local field $\boldsymbol{E}_{\rm loc}(\mathbf{r})$ and the measured or observed field $\boldsymbol{E}'(\mathbf{r})$, the latter being a "screened" version of the former. To find their relationship, we define the susceptibility by $\chi = n^2 - 1$ and require that the polarization $\boldsymbol{P}(\mathbf{r})$ be related to the observed field by the usual relationship $\boldsymbol{P} = \epsilon_0 \chi \boldsymbol{E}'$. Using the Lorentz-Lorenz formula and $\boldsymbol{P} = N\alpha\epsilon_0 \boldsymbol{E}_{\rm loc}$, we find the well-known relationship [598]:

$$\boxed{\boldsymbol{E}_{\rm loc} = \boldsymbol{E}' + \frac{\boldsymbol{P}}{3\epsilon_0}} \tag{14.6.21}$$

From $N\alpha \boldsymbol{E}_{\rm loc} = \boldsymbol{P}/\epsilon_0 = \chi \boldsymbol{E}'$, we have $N\alpha \boldsymbol{E}_1' = \chi \boldsymbol{E}_0'$. Then, the second condition (14.6.18) may be expressed in terms of $\boldsymbol{E}_0'$:

$$\frac{\chi\, \boldsymbol{k}\times(\boldsymbol{E}_0'\times\boldsymbol{k})}{2k_z(k_z'-k_z)} = \boldsymbol{E}_0 \quad\Rightarrow\quad \frac{\boldsymbol{k}\times(\boldsymbol{E}_0'\times\boldsymbol{k})}{k^2} = \frac{2k_z}{k_z'+k_z}\,\boldsymbol{E}_0 \tag{14.6.22}$$

which is identical to Eq. (14.6.10). Thus, the self-consistent solution for $\boldsymbol{E}'(\mathbf{r})$ is identical to that found previously.

Finally, we obtain the reflected field by evaluating Eq. (14.6.13) at points $\mathbf{r}$ to the left of the interface. In this case, there is no $2\boldsymbol{P}/3\epsilon_0$ term in (14.6.14) and we have:

$$\begin{aligned}
\boldsymbol{E}_{\rm rad}(\mathbf{r}) &= N\alpha\,\nabla\times\nabla\times\int_V \boldsymbol{E}_{\rm loc}(\mathbf{r}')\,G(\mathbf{r}-\mathbf{r}')\,dV' = \chi\,\nabla\times\nabla\times\int_V \boldsymbol{E}'(\mathbf{r}')\,G(\mathbf{r}-\mathbf{r}')\,dV' \\
&= \chi\,\nabla\times\nabla\times\boldsymbol{E}_0'\int_V e^{-j\boldsymbol{k}'\cdot\mathbf{r}}\,G(\mathbf{r}-\mathbf{r}')\,dV' = \chi\,\nabla\times\nabla\times\boldsymbol{E}_0'\left[-\frac{e^{-j\boldsymbol{k}_-\cdot\mathbf{r}}}{2k_z(k_z'+k_z)}\right] \\
&= -\frac{\chi\,\boldsymbol{k}_-\times(\boldsymbol{E}_0'\times\boldsymbol{k}_-)}{2k_z(k_z'+k_z)}\,e^{-j\boldsymbol{k}_-\cdot\mathbf{r}} = \frac{k_z-k_z'}{2k_z}\,\frac{\boldsymbol{k}_-\times(\boldsymbol{E}_0'\times\boldsymbol{k}_-)}{k^2}\,e^{-j\boldsymbol{k}_-\cdot\mathbf{r}} = \boldsymbol{E}_{-0}\,e^{-j\boldsymbol{k}_-\cdot\mathbf{r}}
\end{aligned}$$

which agrees with Eq. (14.6.11).

14.7 Radiation Fields

The retarded solutions (14.3.3) for the potentials are quite general and apply to any current and charge distribution. Here, we begin making a number of approximations that are relevant for radiation problems. We are interested in fields that have radiated away from their current sources and are capable of carrying power to large distances from the sources.

The *far-field approximation* assumes that the field point $\mathbf{r}$ is very far from the current source. Here, "far" means much farther than the typical spatial extent of the current distribution, that is, $r \gg r'$. Because r' varies only over the current source we can state this condition as $r \gg l$, where l is the typical extent of the current distribution (for example, for a linear antenna, l is its length.) Fig. 14.7.1 shows this approximation.

As shown in Fig. 14.7.1, at far distances the sides PP′ and PQ of the triangle PQP′ are almost equal. But the side PQ is the difference OP−OQ. Thus, $R \simeq r - \hat{\mathbf{r}}\cdot\mathbf{r}' = r - r'\cos\psi$, where ψ is the angle between the vectors $\mathbf{r}$ and $\mathbf{r}'$.

A better approximation may be obtained with the help of the small-x Taylor series expansion $\sqrt{1+x} \simeq 1 + x/2 - x^2/8$. Expanding R in powers of r'/r, and keeping terms up to second order, we obtain:

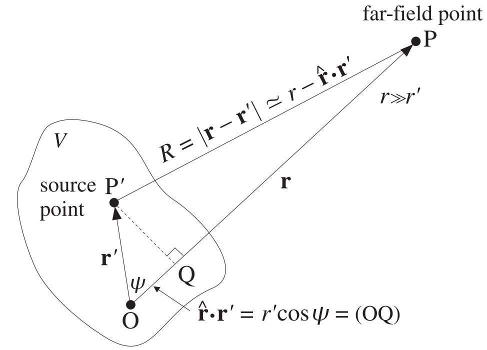


Fig. 14.7.1 Far-field approximation.

$$\begin{aligned}
R = |\mathbf{r}-\mathbf{r}'| &= \sqrt{r^2 - 2rr'\cos\psi + r'^2} = r\sqrt{1 - 2\frac{r'}{r}\cos\psi + \frac{r'^2}{r^2}} \\
&\simeq r\left(1 - \frac{r'}{r}\cos\psi + \frac{r'^2}{2r^2} - \frac{1}{8}\left(-2\frac{r'}{r}\cos\psi + \frac{r'^2}{r^2}\right)^2\right) \\
&\simeq r\left(1 - \frac{r'}{r}\cos\psi + \frac{r'^2}{2r^2} - \frac{r'^2}{2r^2}\cos^2\psi\right)
\end{aligned}$$

or, combining the last two terms:

$$R = r - r'\cos\psi + \frac{r'^2}{2r}\sin^2\psi\,, \qquad \text{for}\quad r \gg r' \tag{14.7.2}$$

Thus, the first-order approximation is $R = r - r'\cos\psi = r - \hat{\mathbf{r}}\cdot\mathbf{r}'$. Using this approximation in the integrands of Eqs. (14.3.1), we have:

$$\varphi(\mathbf{r}) \simeq \int_V \frac{\rho(\mathbf{r}')\,e^{-jk(r-\hat{\mathbf{r}}\cdot\mathbf{r}')}}{4\pi\epsilon(r-\hat{\mathbf{r}}\cdot\mathbf{r}')}\,d^3\mathbf{r}'$$

Replacing $R = r - \hat{\mathbf{r}}\cdot\mathbf{r}' \simeq r$ in the denominator, but not in the exponent, we obtain the far-field approximation to the solution:

$$\varphi(\mathbf{r}) = \frac{e^{-jkr}}{4\pi\epsilon r}\int_V \rho(\mathbf{r}')\,e^{jk\hat{\mathbf{r}}\cdot\mathbf{r}'}\,d^3\mathbf{r}'$$

Because R is approximated differently in the denominator and the exponent, it might be argued that we are not making a consistent approximation. Indeed, for multipole expansions, it is not correct to ignore the $\hat{\mathbf{r}}\cdot\mathbf{r}'$ term from the denominator. However, the procedure is correct for *radiation problems*, and generates those terms that correspond to propagating waves.

What about the second-order approximation terms? We have dropped them from both the exponent and the denominator. Because in the exponent they are multiplied by k, in order to justify dropping them, we must require in addition to $r \gg r'$ that $kr'^2/r \ll 1$, or in terms of the wavelength: $r \gg 2\pi r'^2/\lambda$. Replacing $2r'$ by the typical size l of the current source,† we have $r \gg \pi l^2/2\lambda$. By convention [92], we replace this with $r \gg 2l^2/\lambda$. Thus, we may state the far-field conditions as:

$$r \gg l \qquad \text{and} \qquad r \gg \frac{2l^2}{\lambda} \qquad \text{(far-field conditions)} \tag{14.7.3}$$

These conditions define the so-called far-field or Fraunhofer radiation region. They are easily satisfied for many practical antennas (such as the half-wave dipole) because l is typically of the same order of magnitude as λ, in which case the second condition is essentially equivalent to the first. This happens also when $l > \lambda$. When $l \ll \lambda$, the first condition implies the second.

The distance $r = 2l^2/\lambda$ is by convention [92] the dividing line between the far-field (Fraunhofer) region, and the near-field (Fresnel) region, as shown in Fig. 14.7.2. The far-field region is characterized by the property that the *angular* distribution of radiation is independent of the distance r.

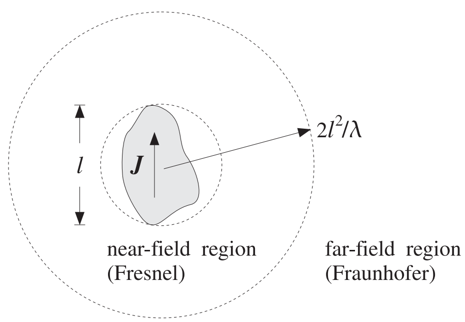


Fig. 14.7.2 Far-field and near-field radiation zones.

Can the first-order term $k\hat{\mathbf{r}}\cdot\mathbf{r}'$ also be ignored from the exponent? This would require that $kr' \ll 1$, or that $r' \ll \lambda$. Thus, it can be ignored for electrically "short" antennas, that is, $l \ll \lambda$, or equivalently in the long wavelength or low-frequency limit. The Hertzian dipole is such an antenna example.

Defining the wavenumber vector $\boldsymbol{k}$ to be in the direction of the field vector $\mathbf{r}$ and having magnitude k, that is, $\boldsymbol{k} = k\hat{\mathbf{r}}$, we may summarize the *far-field approximation* to the retarded single-frequency potentials as follows:

$$\boxed{\begin{aligned} \varphi(\mathbf{r}) &= \frac{e^{-jkr}}{4\pi\epsilon r}\int_V \rho(\mathbf{r}')\,e^{j\boldsymbol{k}\cdot\mathbf{r}'}\,d^3\mathbf{r}' \\ \boldsymbol{A}(\mathbf{r}) &= \frac{\mu e^{-jkr}}{4\pi r}\int_V \boldsymbol{J}(\mathbf{r}')\,e^{j\boldsymbol{k}\cdot\mathbf{r}'}\,d^3\mathbf{r}' \end{aligned}}\,, \qquad \boxed{\boldsymbol{k} = k\hat{\mathbf{r}}} \tag{14.7.4}$$

†We envision a sphere of diameter $2r' = l$ enclosing the antenna structure.

In these expressions, the radial dependence on r has been separated from the angular (θ,ϕ)-dependence, which is given by the integral factors. Since these factors, play an important role in determining the *directional* properties of the radiated fields, we will denote them by the special notation:

$$\begin{aligned} Q(\boldsymbol{k}) &= \int_V \rho(\mathbf{r}')\,e^{j\boldsymbol{k}\cdot\mathbf{r}'}\,d^3\mathbf{r}' \\ \boldsymbol{F}(\boldsymbol{k}) &= \int_V \boldsymbol{J}(\mathbf{r}')\,e^{j\boldsymbol{k}\cdot\mathbf{r}'}\,d^3\mathbf{r}' \qquad \text{(radiation vector)} \end{aligned} \tag{14.7.5}$$

The first is also called the *charge form-factor*, and the second, the *radiation vector*. They are recognized to be the 3-dimensional spatial *Fourier transforms* of the charge and current densities. These quantities depend on ω or k and the directional unit vector $\hat{\mathbf{r}}$ which is completely defined by the spherical coordinate angles θ,ϕ. Therefore, whenever appropriate, we will indicate only the angular dependence in these quantities by writing them as $Q(\theta,\phi)$, $\boldsymbol{F}(\theta,\phi)$. In terms of this new notation, the far-field radiation potentials are:

$$\boxed{\begin{aligned} \varphi(\mathbf{r}) &= \frac{e^{-jkr}}{4\pi\epsilon r}\,Q(\theta,\phi) \\ \boldsymbol{A}(\mathbf{r}) &= \frac{\mu e^{-jkr}}{4\pi r}\,\boldsymbol{F}(\theta,\phi) \end{aligned}} \qquad \text{(radiation potentials)} \tag{14.7.6}$$

14.8 Radial Coordinates

The far-field solutions of Maxwell's equations and the directional patterns of antenna systems are best described in spherical coordinates.

The definitions of cartesian, cylindrical, and spherical coordinate systems are reviewed in Fig. 14.8.1 and are discussed further in Appendix E. The coordinates representing the vector $\mathbf{r}$ are, respectively, (x,y,z), (ρ,ϕ,z), and (r,θ,ϕ) and define orthogonal unit vectors in the corresponding directions, as shown in the figure.

The relationships between coordinate systems can be obtained by viewing the xy-plane and $z\rho$-plane, as shown in Fig. 14.8.2. The relationships between cartesian and cylindrical coordinates are:

$$\boxed{\begin{aligned} x &= \rho\cos\phi \\ y &= \rho\sin\phi \end{aligned}} \qquad \boxed{\begin{aligned} \hat{\boldsymbol{\rho}} &= \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi \\ \hat{\boldsymbol{\phi}} &= -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi \end{aligned}} \tag{14.8.1}$$

Similarly, the relationships of cylindrical to spherical coordinates are:

$$\boxed{\begin{aligned} \rho &= r\sin\theta \\ z &= r\cos\theta \end{aligned}} \qquad \boxed{\begin{aligned} \hat{\mathbf{r}} &= \hat{\mathbf{z}}\cos\theta + \hat{\boldsymbol{\rho}}\sin\theta \\ \hat{\boldsymbol{\theta}} &= -\hat{\mathbf{z}}\sin\theta + \hat{\boldsymbol{\rho}}\cos\theta \end{aligned}} \qquad \boxed{\begin{aligned} \hat{\mathbf{z}} &= \hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta \\ \hat{\boldsymbol{\rho}} &= \hat{\mathbf{r}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta \end{aligned}} \tag{14.8.2}$$

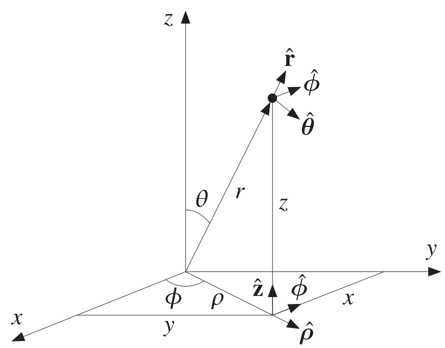


Fig. 14.8.1 Cartesian, cylindrical, and spherical coordinates.

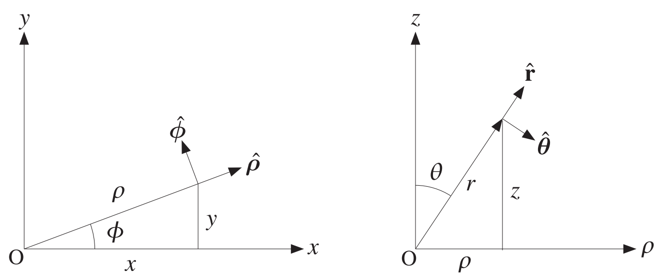


Fig. 14.8.2 Spherical coordinates viewed from xy-plane and $z\rho$-plane.

The relationships between cartesian and spherical coordinates are obtained from (14.8.2) by replacing ρ and $\hat{\boldsymbol{\rho}}$ in terms of Eq. (14.8.1), for example,

$$x = \rho\cos\phi = (r\sin\theta)\cos\phi = r\sin\theta\cos\phi$$

$$\hat{\mathbf{r}} = \hat{\boldsymbol{\rho}}\sin\theta + \hat{\mathbf{z}}\cos\theta = (\hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi)\sin\theta + \hat{\mathbf{z}}\cos\theta$$

The resulting relationships are:

$$\boxed{\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}} \qquad \boxed{\begin{aligned} \hat{\mathbf{r}} &= \hat{\mathbf{x}}\cos\phi\sin\theta + \hat{\mathbf{y}}\sin\phi\sin\theta + \hat{\mathbf{z}}\cos\theta \\ \hat{\boldsymbol{\theta}} &= \hat{\mathbf{x}}\cos\phi\cos\theta + \hat{\mathbf{y}}\sin\phi\cos\theta - \hat{\mathbf{z}}\sin\theta \\ \hat{\boldsymbol{\phi}} &= -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi \end{aligned}} \qquad (14.8.3)$$

Note again that the radial unit vector $\hat{\mathbf{r}}$ is completely determined by the polar and azimuthal angles θ, ϕ. Infinitesimal length increments in each of the spherical unit-vector directions are defined by:

$$dl_r = dr\,, \quad dl_\theta = r\,d\theta\,, \quad dl_\phi = r\sin\theta\,d\phi \qquad \text{(spherical lengths)} \qquad (14.8.4)$$

The gradient operator ∇ in spherical coordinates is:

$$\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial l_r} + \hat{\boldsymbol{\theta}}\frac{\partial}{\partial l_\theta} + \hat{\boldsymbol{\phi}}\frac{\partial}{\partial l_\phi} = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \qquad (14.8.5)$$

The lengths dl_θ and dl_ϕ correspond to infinitesimal displacements in the $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ directions on the surface of a sphere of radius r, as shown in Fig. 14.8.3. The surface element $d\mathbf{S} = \hat{\mathbf{r}}\,dS$ on the sphere is defined by $dS = dl_\theta\,dl_\phi$, or,

$$dS = r^2\sin\theta\,d\theta\,d\phi \qquad (14.8.6)$$

The corresponding infinitesimal solid angle $d\Omega$ subtended by the $d\theta, d\phi$ cone is:

$$\boxed{dS = r^2 d\Omega} \quad \Rightarrow \quad \boxed{d\Omega = \frac{dS}{r^2} = \sin\theta\,d\theta\,d\phi} \qquad (14.8.7)$$

The solid angle subtended by the whole sphere is in units of *steradians*:

$$\Omega_{\text{sphere}} = \int_0^\pi \sin\theta\,d\theta\int_0^{2\pi} d\phi = 4\pi$$

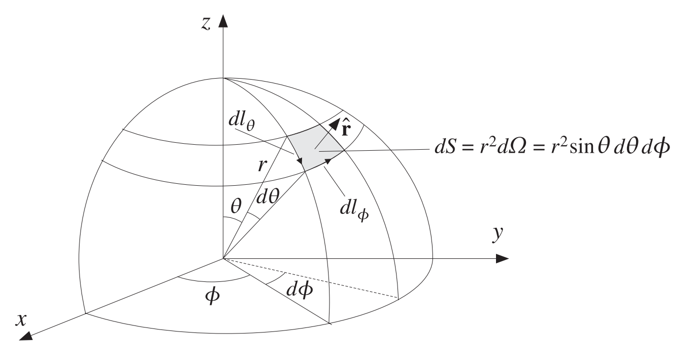


Fig. 14.8.3 Solid angle defined by angles θ, ϕ.

14.9 Radiation Field Approximation

In deriving the field intensities $\boldsymbol{E}$ and $\boldsymbol{H}$ from the far-field potentials (14.7.6), we must make one final approximation and keep only the terms that depend on r like $1/r$, and ignore terms that fall off faster, e.g., like $1/r^2$. We will refer to fields with $1/r$ dependence as *radiation fields.*

The justification for this approximation is shown in Fig. 14.9.1. The power radiated into a solid angle $d\Omega$ will flow through the surface area dS and will be given by $dP = \mathcal{P}_r dS$, where $\mathcal{P}_r$ is the radial component of the Poynting vector. Replacing dS in terms of the solid angle and $\mathcal{P}_r$ in terms of the squared electric field, we have:

$$dP = \mathcal{P}_r dS = \left(\frac{1}{2\eta}|E|^2\right)(r^2 d\Omega)$$

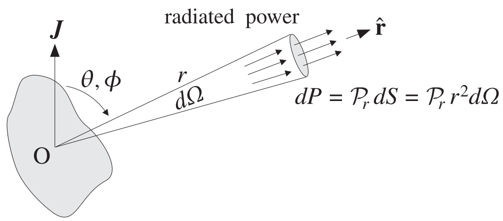


Fig. 14.9.1 Power radiated into solid angle $d\Omega$.

Thus, if the amount of power in the solid angle $d\Omega$ is to propagate away without attenuation with distance r, then the electric field must be such that $|E|^2r^2 \sim$ const, or that $|E| \sim 1/r$; similarly, $|H| \sim 1/r$. Any terms in E, H that fall off faster than $1/r$ will not be capable of radiating power to large distances from their current sources.

14.10 Computing the Radiation Fields

At far distances from the localized current $\boldsymbol{J}$, the radiation fields can be obtained from Eqs. (14.3.9) by using the radiation vector potential $\boldsymbol{A}$ of Eq. (14.7.6). In computing the curl of $\boldsymbol{A}$, we may ignore any terms that fall off faster than $1/r$:

$$\nabla\times\boldsymbol{A} = \nabla\times\left(\frac{\mu e^{-jkr}}{4\pi r}\boldsymbol{F}\right) = \left(\hat{\mathbf{r}}\frac{\partial}{\partial r} + \text{angular derivatives}\right)\times\left(\frac{\mu e^{-jkr}}{4\pi r}\boldsymbol{F}\right)$$

$$= -jk\,(\hat{\mathbf{r}}\times\boldsymbol{F})\left(\frac{\mu e^{-jkr}}{4\pi r}\right) + O\left(\frac{1}{r^2}\right) = -j\,\boldsymbol{k}\times\boldsymbol{A} + O\left(\frac{1}{r^2}\right)$$

The "angular derivatives" arise from the θ, ϕ derivatives in the gradient as per Eq. (14.8.5). These derivatives act on $\boldsymbol{F}(\theta,\phi)$, but because they already have a $1/r$ factor in them and the rest of $\boldsymbol{A}$ has another $1/r$ factor, these terms will go down like $1/r^2$. Similarly, when we compute the derivative $\partial_r[e^{-jkr}/r]$ we may keep only the derivative of the numerator because the rest goes down like $1/r^2$.

Thus, we arrive at the useful *rule* that to order $1/r$, the gradient operator ∇, *whenever* it acts on a function of the form $f(\theta,\phi)\,e^{-jkr}/r$, can be replaced by:

$$\boxed{\nabla \longrightarrow -j\,\boldsymbol{k} = -jk\,\hat{\mathbf{r}}} \tag{14.10.1}$$

Applying the rule again, we have:

$$\nabla\times(\nabla\times\boldsymbol{A}) = -j\,\boldsymbol{k}\times(-j\,\boldsymbol{k}\times\boldsymbol{A}) = (\boldsymbol{k}\times\boldsymbol{A})\times\boldsymbol{k} = k^2(\hat{\mathbf{r}}\times\boldsymbol{A})\times\hat{\mathbf{r}} = \omega^2\mu\epsilon(\hat{\mathbf{r}}\times\boldsymbol{A})\times\hat{\mathbf{r}}$$

Noting that $\omega\mu = ck\mu = k\sqrt{\mu/\epsilon} = k\eta$ and using Eq. (14.3.9), we finally find:

$$\boxed{\begin{aligned}\boldsymbol{E} &= -jk\eta\,\frac{e^{-jkr}}{4\pi r}\,(\hat{\mathbf{r}}\times\boldsymbol{F})\times\hat{\mathbf{r}}\\ \boldsymbol{H} &= -jk\,\frac{e^{-jkr}}{4\pi r}\,\hat{\mathbf{r}}\times\boldsymbol{F}\end{aligned}}\qquad\text{(radiation fields)} \tag{14.10.2}$$

Moreover, we recognize that:

$$\boxed{\boldsymbol{E} = \eta\boldsymbol{H}\times\hat{\mathbf{r}},\quad \boldsymbol{H} = \frac{1}{\eta}\hat{\mathbf{r}}\times\boldsymbol{E}}\qquad\text{and}\qquad \frac{|\boldsymbol{E}|}{|\boldsymbol{H}|} = \eta \tag{14.10.3}$$

We note the similarity to uniform plane waves and emphasize the following properties:

1. $\{\boldsymbol{E}, \boldsymbol{H}, \hat{\mathbf{r}}\}$ form a right-handed vector system.
2. $\boldsymbol{E}$ is always parallel to the transverse part $\boldsymbol{F}_\perp$ of the radiation vector $\boldsymbol{F}$.
3. $\boldsymbol{H}$ is always perpendicular to the radiation vector $\boldsymbol{F}$.
4. dc current sources $(\omega = k = 0)$ will not radiate.

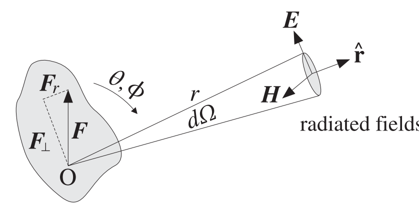


Fig. 14.10.1 Electric and magnetic fields radiated by a current source.

Figure 14.10.1 illustrates some of these remarks. The radiation vector may be decomposed in general into a radial part $\boldsymbol{F}_r = \hat{\mathbf{r}}F_r$ and a transverse part $\boldsymbol{F}_\perp$. In fact, this decomposition is obtained from the identity:

$$\boldsymbol{F} = \hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\boldsymbol{F}) + (\hat{\mathbf{r}}\times\boldsymbol{F})\times\hat{\mathbf{r}} = \hat{\mathbf{r}}F_r + \boldsymbol{F}_\perp$$

Resolving $\boldsymbol{F}$ along the spherical coordinate unit vectors, we have:

$$\boldsymbol{F} = \hat{\mathbf{r}}F_r + \hat{\boldsymbol{\theta}}F_\theta + \hat{\boldsymbol{\phi}}F_\phi$$

$$\hat{\mathbf{r}}\times\boldsymbol{F} = \hat{\boldsymbol{\phi}}F_\theta - \hat{\boldsymbol{\theta}}F_\phi$$

$$\boldsymbol{F}_\perp = (\hat{\mathbf{r}}\times\boldsymbol{F})\times\hat{\mathbf{r}} = \hat{\boldsymbol{\theta}}F_\theta + \hat{\boldsymbol{\phi}}F_\phi$$

Thus, only F_θ and F_ϕ contribute to the fields:

$$\boxed{\begin{aligned}\boldsymbol{E} &= -jk\eta\,\frac{e^{-jkr}}{4\pi r}\,[\hat{\boldsymbol{\theta}}F_\theta + \hat{\boldsymbol{\phi}}F_\phi]\\ \boldsymbol{H} &= -jk\,\frac{e^{-jkr}}{4\pi r}\,[\hat{\boldsymbol{\phi}}F_\theta - \hat{\boldsymbol{\theta}}F_\phi]\end{aligned}}\qquad\text{(radiation fields)} \tag{14.10.4}$$

Recognizing that $\hat{\mathbf{r}}\times\boldsymbol{F} = \hat{\mathbf{r}}\times\boldsymbol{F}_\perp$, we can also write compactly:

$$\boxed{\begin{aligned}\boldsymbol{E} &= -jk\eta\,\frac{e^{-jkr}}{4\pi r}\,\boldsymbol{F}_\perp\\ \boldsymbol{H} &= -jk\,\frac{e^{-jkr}}{4\pi r}\,\hat{\mathbf{r}}\times\boldsymbol{F}_\perp\end{aligned}}\qquad\text{(radiation fields)} \tag{14.10.5}$$

In general, the radiation vector will have both F_θ and F_ϕ components, depending on the nature of the current distribution $\boldsymbol{J}$. However, in practice there are three important cases that stand out:

1. Only F_θ is present. This includes all *linear antennas* and arrays. The z-axis is oriented in the direction of the antenna, so that the radiation vector only has r and θ components.
2. Only F_ϕ is present. This includes *loop antennas* with the xy-plane chosen as the plane of the loop.
3. Both F_θ and F_ϕ are present, but they are carefully chosen to have the phase relationship $F_\phi = \pm jF_\theta$, so that the resulting electric field will be *circularly polarized.* This includes *helical antennas* used in space communications.

14.11 Problems

14.1 First, prove the differential identity:

$$\nabla'\cdot\big[\boldsymbol{J}(\mathbf{r}')e^{j\boldsymbol{k}\cdot\mathbf{r}'}\big] = j\,\boldsymbol{k}\cdot\boldsymbol{J}(\mathbf{r}')e^{j\boldsymbol{k}\cdot\mathbf{r}'} - j\omega\rho(\mathbf{r}')e^{j\boldsymbol{k}\cdot\mathbf{r}'}$$

Then, prove the integral identity:

$$\boldsymbol{k}\cdot\int_V \boldsymbol{J}(\mathbf{r}')e^{j\boldsymbol{k}\cdot\mathbf{r}'}\,d^3\mathbf{r}' = \omega\int_V \rho(\mathbf{r}')e^{j\boldsymbol{k}\cdot\mathbf{r}'}\,d^3\mathbf{r}'$$

Assume that the charge and current densities are localized within the finite volume V. Finally, show that the charge form-factor Q and radiation vector $\boldsymbol{F}$ are related by:

$$\hat{\mathbf{r}}\cdot\boldsymbol{F} = cQ$$

14.2 Using similar techniques as in the previous problem, prove the following general property, valid for any scalar function $g(\mathbf{r})$, where V is the volume over which $\boldsymbol{J},\rho$ are non-zero:

$$\int_V \boldsymbol{J}(\mathbf{r}')\cdot\nabla' g(\mathbf{r}')\,d^3\mathbf{r}' = j\omega\int_V g(\mathbf{r}')\rho(\mathbf{r}')\,d^3\mathbf{r}'$$

14.3 It is possible to obtain the fields generated by the source densities $\rho,\boldsymbol{J}$ by working directly with Maxwell's equations without introducing the scalar and vector potentials $\phi,\boldsymbol{A}$. Start with the monochromatic Maxwell's equations

$$\nabla\times\boldsymbol{E} = -j\omega\mu\boldsymbol{H},\quad \nabla\times\boldsymbol{H} = \boldsymbol{J}+j\omega\epsilon\boldsymbol{E},\quad \nabla\cdot\boldsymbol{E} = \frac{1}{\epsilon}\rho,\quad \nabla\cdot\boldsymbol{H} = 0$$

Show that $\boldsymbol{E},\boldsymbol{H}$ satisfy the following Helmholtz equations:

$$(\nabla^2+k^2)\boldsymbol{E} = j\omega\mu\boldsymbol{J}+\frac{1}{\epsilon}\nabla\rho,\quad (\nabla^2+k^2)\boldsymbol{H} = -\nabla\times\boldsymbol{J}$$

Show that their solutions are obtained with the help of the Green's function (14.3.4):

$$\boldsymbol{E} = \int_V\big[-j\omega\mu\boldsymbol{J}G - \frac{1}{\epsilon}(\nabla'\rho)G\big]\,dV'$$

$$\boldsymbol{H} = \int_V\big[\nabla'\times\boldsymbol{J}\big]G\,dV'$$

Although these expressions and Eqs. (14.3.10) look slightly different, they are equivalent. Explain in what sense this is true.

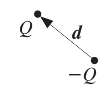
14.4 The electric and magnetic dipole moments of charge and current volume distributions $\rho,\boldsymbol{J}$ are defined by:

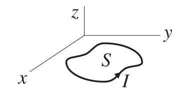
$$\mathbf{p} = \int_V \mathbf{r}\,\rho(\mathbf{r})\,dV\,,\quad \mathbf{m} = \frac{1}{2}\int_V \mathbf{r}\times\boldsymbol{J}(\mathbf{r})\,dV$$

Using these definitions and the integral property of Eq. (C.41) of Appendix C, show that for two charges $\pm Q$ separated by distance $\mathbf{d}$, and for a current I flowing on a closed planar loop of arbitrary shape and area S lying on the xy-plane, the quantities $\mathbf{p},\mathbf{m}$ are given by:

$$\mathbf{p} = Q\,\mathbf{d}$$

$$\mathbf{m} = \hat{\mathbf{z}}\,I\,S$$





14.5 By performing an inverse Fourier time transform on Eq. (14.5.5), show that the fields produced by an arbitrary time-varying dipole at the origin, $\boldsymbol{P}(\mathbf{r},t) = \mathbf{p}(t)\,\delta^{(3)}(\mathbf{r})$, are given by:

$$\boldsymbol{E}(\mathbf{r},t) = \frac{1}{\epsilon_0}\Big(\frac{1}{c_0}\frac{\partial}{\partial t}+\frac{1}{r}\Big)\left[\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{p}(t_r)) - \mathbf{p}(t_r)}{r}\right]\frac{1}{4\pi r} - \frac{1}{\epsilon_0 c_0^2}\,\hat{\mathbf{r}}\times(\ddot{\mathbf{p}}(t_r)\times\hat{\mathbf{r}})\,\frac{1}{4\pi r}$$

$$\boldsymbol{H}(\mathbf{r},t) = \frac{\partial}{\partial t}\Big(\frac{1}{c_0}\frac{\partial}{\partial t}+\frac{1}{r}\Big)(\mathbf{p}(t_r)\times\hat{\mathbf{r}})\,\frac{1}{4\pi r}$$

where $t_r = t - r/c_0$ is the retarded time and the time-derivatives act only on $\mathbf{p}(t_r)$. Show also that the radiated fields are (with $\eta_0 = \sqrt{\mu_0/\epsilon_0}$):

$$\boldsymbol{E}_{\text{rad}}(\mathbf{r},t) = \mu_0\,\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\ddot{\mathbf{p}}(t_r))\,\frac{1}{4\pi r} = \eta_0\,\boldsymbol{H}_{\text{rad}}(\mathbf{r},t)\times\hat{\mathbf{r}}$$

$$\boldsymbol{H}_{\text{rad}}(\mathbf{r},t) = \frac{\mu_0}{\eta_0}(\ddot{\mathbf{p}}(t_r)\times\hat{\mathbf{r}})\,\frac{1}{4\pi r}$$

14.6 Assume that the dipole of the previous problem is along the z-direction, $\mathbf{p}(t) = \hat{\mathbf{z}}\,p(t)$. Integrating the Poynting vector $\boldsymbol{\mathcal{P}} = \boldsymbol{E}_{\text{rad}}\times\boldsymbol{H}_{\text{rad}}$ over a sphere of radius r, show that the total radiated power from the dipole is given by:

$$P_{\text{rad}}(r,t) = \frac{\eta_0}{6\pi c_0^2}\,\ddot{p}^2(t_r)$$

14.7 Define a 3×3 matrix $J(\mathbf{a})$ such that the operation $J(\mathbf{a})\mathbf{b}$ represents the cross-product $\mathbf{a}\times\mathbf{b}$. Show that:

$$J(\mathbf{a}) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

Show that $J(\mathbf{a})$ is a rank-2 matrix with eigenvalues $\lambda = 0$ and $\lambda = \pm j|\mathbf{a}|$, where $\mathbf{a}$ is assumed to be real-valued. Show that the eigenvectors corresponding to the non-zero eigenvalues are given by $\mathbf{e} = \hat{\mathbf{f}}\mp j\,\hat{\mathbf{g}}$, where $\hat{\mathbf{f}},\hat{\mathbf{g}}$ are real-valued unit vectors such that $\{\hat{\mathbf{f}},\hat{\mathbf{g}},\hat{\mathbf{a}}\}$ is a right-handed vector system (like $\{\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}}\}$), here, $\hat{\mathbf{a}} = \mathbf{a}/|\mathbf{a}|$. Show that $\mathbf{e}\cdot\mathbf{e} = 0$ and $\mathbf{e}^*\cdot\mathbf{e} = 2$.

A radiator consists of electric and magnetic dipoles $\mathbf{p},\mathbf{m}$ placed at the origin. Assuming harmonic time dependence and adding the radiation fields of Eqs. (14.5.6) and (14.5.10), show that the total radiated fields can be expressed in terms of the 6×6 matrix operation:

$$\begin{bmatrix} \boldsymbol{E}(\mathbf{r}) \\ \eta_0\boldsymbol{H}(\mathbf{r}) \end{bmatrix} = -\eta_0\,k^2\,\frac{e^{-jkr}}{4\pi r}\begin{bmatrix} J^2(\hat{\mathbf{r}}) & J(\hat{\mathbf{r}}) \\ -J(\hat{\mathbf{r}}) & J^2(\hat{\mathbf{r}}) \end{bmatrix}\begin{bmatrix} c_0\,\mathbf{p} \\ \mathbf{m} \end{bmatrix}$$

Show that $J(\hat{\mathbf{r}})$ satisfies the matrix equation $J^3(\hat{\mathbf{r}})+J(\hat{\mathbf{r}})=0$. Moreover, show that its eigenvalues are $\lambda = 0$ and $\lambda = \pm j$ and that the eigenvectors belonging to the two nonzero eigenvalues are given in terms of the polar unit vectors by $\mathbf{e} = \hat{\boldsymbol{\theta}} \mp j\,\hat{\boldsymbol{\phi}}$.

Because the matrix $J(\hat{\mathbf{r}})$ is rank-defective, so is the above 6×6 matrix, reflecting the fact that the radiation fields can only have two polarization states. However, it has been shown recently [1003] that in a multiple-scattering environment, such as wireless propagation in cities, the corresponding 6×6 matrix becomes a full-rank matrix (rank 6) allowing the tripling of the channel capacity over the standard dual-polarization transmission.