

MIDTERM FORMULA SHEET

VECTOR IDENTITIES

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{C} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{C} \times \vec{A})$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

$$\nabla(\Phi + \Psi) = \nabla\Phi + \nabla\Psi$$

$$\nabla \cdot (\vec{A} + \vec{B}) = \nabla \cdot \vec{A} + \nabla \cdot \vec{B}$$

$$\nabla \times (\vec{A} + \vec{B}) = \nabla \times \vec{A} + \nabla \times \vec{B}$$

$$\nabla(\Phi\Psi) = \Phi\nabla\Psi + \Psi\nabla\Phi$$

$$\nabla \left(\frac{\Phi}{\Psi} \right) = \frac{\Psi\nabla\Phi - \Phi\nabla\Psi}{\Psi^2}$$

$$\nabla\Phi^n = n\Phi^{n-1}\nabla\Phi$$

$$\nabla \cdot (\Phi\vec{A}) = \vec{A} \cdot \nabla\Phi + \Phi\nabla \cdot \vec{A}$$

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot \nabla \times \vec{A} - \vec{A} \cdot \nabla \times \vec{B}$$

$$\nabla \times (\Phi\vec{A}) = \nabla\Phi \times \vec{A} + \Phi\nabla \times \vec{A}$$

$$\nabla \times (\vec{A} \times \vec{B}) = \vec{A}\nabla \cdot \vec{B} - \vec{B}\nabla \cdot \vec{A} + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B}$$

$$\nabla(\vec{A} \cdot \vec{B}) = \vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A}) + (\vec{A} \cdot \nabla)\vec{B} + (\vec{B} \cdot \nabla)\vec{A}$$

$$\nabla \cdot \nabla\Phi = \nabla^2\Phi$$

$$\nabla \cdot \nabla \times \vec{A} = 0$$

$$\nabla \times \nabla\Phi = 0$$

$$\nabla \times \nabla \times \vec{A} = \nabla\nabla \cdot \vec{A} - \nabla^2\vec{A}$$

VECTOR INTEGRAL THEOREMS

$$\iiint_V (\nabla \cdot \vec{A}) dv = \oiint_{S_{[v]}} \vec{A} \cdot d\vec{s} \quad (\text{Divergence theorem, Gauss identity})$$

$$\iint_S (\nabla \times \vec{A}) \cdot d\vec{s} = \oint_{C_{[s]}} \vec{A} \cdot d\vec{l} \quad (\text{Curl theorem 1, Stokes' theorem})$$

$$\iiint_V (\nabla \times \vec{A}) dv = \oiint_{S_{[v]}} d\vec{s} \times \vec{A} \equiv \oiint_{S_{[v]}} (\hat{n} \times \vec{A}) ds \quad (\text{Curl theorem 2})$$

SOME INTEGRALS OFTEN MET IN EM PROBLEMS

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int \frac{1}{(a^2 \pm x^2)^{3/2}} dx = \pm \frac{x}{a^2 \sqrt{a^2 \pm x^2}} + C$$

$$\int \frac{x}{(a^2 + x^2)^{3/2}} dx = -\frac{1}{\sqrt{a^2 + x^2}} + C$$

$$\int \frac{x^2}{(a^2 + x^2)^{3/2}} dx = -\frac{x}{\sqrt{a^2 + x^2}} + \ln(x + \sqrt{a^2 + x^2}) + C$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \arctan \frac{x}{a} + C$$

$$\int \frac{1}{x^2 - a^2} dx = \begin{cases} \frac{1}{2a} \ln \left(\frac{a-x}{a+x} \right) = -\frac{1}{a} \operatorname{arctanh} \left(\frac{x}{a} \right), & |x| < a \\ -\frac{1}{a} \operatorname{arccoth} \left(\frac{x}{a} \right), & |x| > a \end{cases}$$

$$\int \frac{x}{a^2 + x^2} dx = \frac{1}{2} \ln(a^2 + x^2) + C$$

$$\int \frac{x}{\sqrt{a^2 + x^2}} dx = \sqrt{a^2 + x^2} + C$$

$$\int \frac{1}{\sqrt{a^2 + x^2}} dx = \ln(x + \sqrt{a^2 + x^2}) + C$$

$$\int \frac{1}{x\sqrt{a^2 + x^2}} dx = -\frac{1}{a} \ln \left(\frac{a + \sqrt{a^2 + x^2}}{x} \right) + C$$

$$\int x \sin(ax) dx = \frac{1}{a^2} [\sin(ax) - ax \cos(ax)]$$

$$\int x \cos(ax) dx = \frac{1}{a^2} [\cos(ax) + ax \sin(ax)]$$

MIDTERM FORMULA SHEET

$$\int \frac{dx}{(ax^2 + b)\sqrt{fx^2 + g}} = \frac{1}{\sqrt{b}\sqrt{ag - bf}} \arctan \left(\frac{x\sqrt{ag - bf}}{\sqrt{b}\sqrt{fx^2 + g}} \right), (ag > bf)$$

$$\int \tan x dx = -\ln |\cos x| + C, x \neq (2k+1)\frac{\pi}{2}$$

$$\int \cot x dx = \ln |\sin x| + C, x \neq 2k\pi$$

$$\int \frac{1}{\sin x} dx = \ln \left| \tan \left(\frac{x}{2} \right) \right| + C$$

$$\int \frac{1}{\cos x} dx = \ln \left| \tan \left(\frac{x}{2} + \frac{\pi}{4} \right) \right| + C$$

SOME USEFUL DEFINITE INTEGRALS

$$\int_0^{2\pi} \sin mx \cdot \sin nx dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \neq 0 \end{cases}$$

$$\int_0^{2\pi} \cos mx \cdot \cos nx dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \neq 0 \end{cases}$$

$$\int_0^{2\pi} \sin mx \cdot \cos nx dx = 0$$

$$\int_0^{\pi} \sin mx \cdot \sin nx dx = \begin{cases} 0, & m \neq n \\ \pi/2, & m = n \neq 0 \end{cases}$$

$$\int_0^{\pi} \cos mx \cdot \cos nx dx = \begin{cases} 0, & m \neq n \\ \pi/2, & m = n \neq 0 \end{cases}$$

$$\int_0^{\pi} \sin mx \cdot \cos nx dx = \begin{cases} 0, & m+n = \text{even number} \\ \frac{2m}{m^2 - n^2}, & m+n = \text{odd number} \end{cases}$$

$$\int_0^{\pi} \frac{(a - b \cos x)}{(a^2 + b^2 - 2ab \cos x)} dx = \begin{cases} \frac{\pi}{a}, & a > b > 0 \\ 0, & b > a > 0 \end{cases}$$

COORDINATE TRANSFORMATIONS

Rectangular ↔ Cylindrical

$$\begin{cases} x = r \cos \phi \\ y = r \sin \phi \\ z = z \end{cases} \quad \begin{cases} r = \sqrt{x^2 + y^2} \\ \phi = \arctan \left(\frac{y}{x} \right) \\ z = z \end{cases}$$

Rectangular ↔ Spherical

$$\begin{cases} x = R \sin \theta \cos \phi \\ y = R \sin \theta \sin \phi \\ z = R \cos \theta \end{cases} \quad \begin{cases} R = \sqrt{x^2 + y^2 + z^2} \\ \theta = \arccos \left(\frac{z}{\sqrt{x^2 + y^2 + z^2}} \right) \\ \phi = \arctan(y/x) \end{cases}$$

Cylindrical ↔ Spherical

$$\begin{cases} r = R \sin \theta \\ \phi = \phi \\ z = R \cos \theta \end{cases} \quad \begin{cases} R = \sqrt{r^2 + z^2} \\ \theta = \arccos \left(\frac{z}{\sqrt{r^2 + z^2}} \right) \end{cases}$$

VECTOR TRANSFORMATIONS

Rectangular Components ↔ Cylindrical Components

$$\begin{cases} a_x = a_r \cos \phi - a_\phi \sin \phi \\ a_y = a_r \sin \phi + a_\phi \cos \phi \\ a_z = a_z \end{cases} \quad \begin{cases} a_r = a_x \cos \phi + a_y \sin \phi \\ a_\phi = -a_x \sin \phi + a_y \cos \phi \\ a_z = a_z \end{cases}$$

Note: ϕ is the position angle of the point at which the vector exists.

Rectangular Components ↔ Spherical Components

$$\begin{cases} a_x = a_R \sin \theta \cos \phi + a_\theta \cos \theta \cos \phi - a_\phi \sin \phi \\ a_y = a_R \sin \theta \sin \phi + a_\theta \cos \theta \sin \phi + a_\phi \cos \phi \\ a_z = a_R \cos \theta - a_\theta \sin \theta \end{cases}$$

$$\begin{cases} a_R = a_x \sin \theta \cos \phi + a_y \sin \theta \sin \phi + a_z \cos \theta \\ a_\theta = a_x \cos \theta \cos \phi + a_y \cos \theta \sin \phi - a_z \sin \theta \\ a_\phi = -a_x \sin \phi + a_y \cos \phi \end{cases}$$

Note: ϕ and θ are the position angles of the point at which the vector exists.

MIDTERM FORMULA SHEET

Cylindrical Components ↔ Spherical Components

$$\begin{cases} a_r = a_R \sin \theta + a_\theta \cos \theta \\ a_\phi = a_\phi \\ a_z = a_R \cos \theta - a_\theta \sin \theta \end{cases} \quad \begin{cases} a_R = a_r \sin \theta + a_z \cos \theta \\ a_\theta = a_r \cos \theta - a_z \sin \theta \\ a_\phi = a_\phi \end{cases}$$

Note: θ is the position angle of the point at which the vector exists.

DERIVATIVES OF ELEMENTARY FUNCTIONS

$(const.)' = 0$	$(\arctan x)' = \frac{1}{1+x^2}$
$(x)' = 1$	$(\operatorname{arccot} x)' = -\frac{1}{1+x^2}$
$(x^k)' = kx^{k-1}$	$(\sinh x)' = \cosh x$
$(e^x)' = e^x$	$(\cosh x)' = \sinh x$
$(a^x)' = a^x \ln a$	$(\tanh x)' = \frac{1}{\cosh^2 x} = 1 - \tanh^2 x$
$(\ln x)' = \frac{1}{x}$	$(\operatorname{coth} x)' = -\frac{1}{\sinh^2 x} = 1 - \operatorname{coth}^2 x$
$(\log_a x)' = \frac{1}{x \ln a}, a \neq 1, x > 0$	$(\operatorname{arcsinh} x)' = \frac{1}{\sqrt{1+x^2}}$
$(\sin x)' = \cos x$	$(\operatorname{arccosh} x)' = \pm \frac{1}{\sqrt{x^2-1}}, x > 1$
$(\cos x)' = -\sin x$	$(\operatorname{arctanh} x)' = \frac{1}{1-x^2}, x < 1$
$(\tan x)' = \frac{1}{\cos^2 x}, x \neq (2k+1)\pi$	$(\operatorname{arccoth} x)' = \frac{1}{1-x^2}, x > 1$
$(\cot x)' = -\frac{1}{\sin^2 x}, x \neq k\pi$	
$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}, x < 1$	
$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}, x < 1$	

DIFFERENTIAL OPERATORS

Rectangular Coordinates

$$\nabla \Phi = \hat{x} \frac{\partial \Phi}{\partial x} + \hat{y} \frac{\partial \Phi}{\partial y} + \hat{z} \frac{\partial \Phi}{\partial z}$$

$$\nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\nabla \times \vec{F} = \hat{x} \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + \hat{y} \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) + \hat{z} \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$$

$$\nabla \cdot (\nabla \Phi) \equiv \nabla^2 \Phi \equiv \Delta \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

$$\nabla^2 \vec{F} = \hat{x} \nabla^2 F_x + \hat{y} \nabla^2 F_y + \hat{z} \nabla^2 F_z$$

Cylindrical Coordinates

$$\nabla \Phi = \frac{\partial \Phi}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial \Phi}{\partial \phi} \hat{\phi} + \frac{\partial \Phi}{\partial z} \hat{z}$$

$$\nabla \cdot \vec{F} = \frac{1}{r} \frac{\partial}{\partial r} (r F_r) + \frac{1}{r} \frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}$$

$$\nabla \times \vec{F} = \hat{r} \left(\frac{1}{r} \frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z} \right) + \hat{\phi} \left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r} \right) + \hat{z} \left(\frac{1}{r} \frac{\partial (r F_\phi)}{\partial r} - \frac{1}{r} \frac{\partial F_r}{\partial \phi} \right)$$

$$\nabla \cdot (\nabla \Phi) \equiv \nabla^2 \Phi \equiv \Delta \Phi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

$$\begin{aligned} \nabla^2 \vec{A} = & \hat{r} \left(\frac{\partial^2 A_r}{\partial r^2} + \frac{1}{r} \frac{\partial A_r}{\partial r} - \frac{A_r}{r^2} + \frac{1}{r^2} \frac{\partial^2 A_r}{\partial \phi^2} - \frac{2}{r^2} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial^2 A_r}{\partial z^2} \right) + \\ & \hat{\phi} \left(\frac{\partial^2 A_\phi}{\partial r^2} + \frac{1}{r} \frac{\partial A_\phi}{\partial r} - \frac{A_\phi}{r^2} + \frac{1}{r^2} \frac{\partial^2 A_\phi}{\partial \phi^2} + \frac{2}{r^2} \frac{\partial A_r}{\partial \phi} + \frac{\partial^2 A_\phi}{\partial z^2} \right) + \\ & \hat{z} \left(\frac{\partial^2 A_z}{\partial r^2} + \frac{1}{r} \frac{\partial A_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 A_z}{\partial \phi^2} + \frac{\partial^2 A_z}{\partial z^2} \right) \end{aligned}$$

Spherical Coordinates

$$\nabla \Phi = \frac{\partial \Phi}{\partial R} \hat{R} + \frac{1}{R} \frac{\partial \Phi}{\partial \theta} \hat{\theta} + \frac{1}{R \sin \theta} \frac{\partial \Phi}{\partial \phi} \hat{\phi}$$

$$\nabla \cdot \vec{F} = \frac{1}{R^2} \frac{\partial}{\partial R} (R^2 F_R) + \frac{1}{R \sin \theta} \frac{\partial}{\partial \theta} (F_\theta \sin \theta) + \frac{1}{R \sin \theta} \frac{\partial F_\phi}{\partial \phi}$$

MIDTERM FORMULA SHEET

$$\begin{aligned}\nabla \times \vec{A} &= \hat{R} \frac{1}{R \sin \theta} \left[\frac{\partial}{\partial \theta} (A_\varphi \sin \theta) - \frac{\partial A_\theta}{\partial \varphi} \right] + \\ &\hat{\theta} \frac{1}{R} \left[\frac{1}{\sin \theta} \frac{\partial A_R}{\partial \varphi} - \frac{\partial}{\partial R} (R A_\varphi) \right] + \\ &\hat{\varphi} \frac{1}{R} \left[\frac{\partial}{\partial R} (R A_\theta) - \frac{\partial A_R}{\partial \theta} \right] \\ \nabla^2 \Phi &= \frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \frac{\partial \Phi}{\partial R} \right) + \frac{1}{R^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \Phi}{\partial \theta} \right) + \frac{1}{R^2 \sin^2 \theta} \frac{\partial^2 \Phi}{\partial \varphi^2} \\ \nabla^2 \vec{A} &= \hat{R} \left(\frac{\partial^2 A_R}{\partial R^2} + \frac{2}{R} \frac{\partial A_R}{\partial R} - \frac{2}{R^2} A_R + \frac{1}{R^2} \frac{\partial^2 A_R}{\partial \theta^2} + \frac{\cot \theta}{R^2} \frac{\partial A_R}{\partial \theta} + \right. \\ &\left. \frac{1}{R^2 \sin^2 \theta} \frac{\partial^2 A_R}{\partial \varphi^2} - \frac{2}{R^2} \frac{\partial A_\theta}{\partial \theta} - \frac{2 \cot \theta}{R^2} A_\theta - \frac{2}{R^2 \sin \theta} \frac{\partial A_\varphi}{\partial \varphi} \right) + \\ &\hat{\theta} \left(\frac{\partial^2 A_\theta}{\partial R^2} + \frac{2}{R} \frac{\partial A_\theta}{\partial R} - \frac{A_\theta}{R^2 \sin^2 \theta} + \frac{1}{R^2} \frac{\partial^2 A_\theta}{\partial \theta^2} + \frac{\cot \theta}{R^2} \frac{\partial A_\theta}{\partial \theta} + \right. \\ &\left. \frac{1}{R^2 \sin^2 \theta} \frac{\partial^2 A_\theta}{\partial \varphi^2} + \frac{2}{R^2} \frac{\partial A_R}{\partial \theta} - \frac{2 \cot \theta}{R^2 \sin \theta} \frac{\partial A_\varphi}{\partial \varphi} \right) + \\ &\hat{\varphi} \left(\frac{\partial^2 A_\varphi}{\partial R^2} + \frac{2}{R} \frac{\partial A_\varphi}{\partial R} - \frac{A_\varphi}{R^2 \sin^2 \theta} + \frac{1}{R^2} \frac{\partial^2 A_\varphi}{\partial \theta^2} + \frac{\cot \theta}{R^2} \frac{\partial A_\varphi}{\partial \theta} + \right. \\ &\left. \frac{1}{R^2 \sin^2 \theta} \frac{\partial^2 A_\varphi}{\partial \varphi^2} + \frac{2}{R^2 \sin \theta} \frac{\partial A_R}{\partial \varphi} + \frac{2 \cot \theta}{R^2 \sin \theta} \frac{\partial A_\theta}{\partial \varphi} \right)\end{aligned}$$

$$\begin{aligned}d\vec{l} &= \hat{R}dR + \hat{\theta}Rd\theta + \hat{\varphi}R \sin \theta d\varphi ; \\ d\vec{s} &= \hat{R}R^2 \sin \theta d\theta d\varphi + \hat{\theta}R \sin \theta dR d\varphi + \hat{\varphi}RdRd\theta ; \\ dv &= R^2 \sin \theta dRd\theta d\varphi\end{aligned}$$

ELECTROMAGNETIC EQUATIONS

Coaxial line

$$C_1 = \frac{2\pi\epsilon}{\ln(b/a)}, \text{ F/m}; \quad L_1 = \frac{\mu_0}{2\pi} \ln\left(\frac{b}{a}\right) + \frac{\mu_0}{8\pi}, \text{ H/m}$$

Twin-lead line

$$C_1 = \frac{\pi\epsilon}{\ln\left(\frac{h}{r} + \sqrt{\left(\frac{h}{r}\right)^2 - 1}\right)} \text{ F/m}; \quad L_1 = \frac{\mu}{\pi} \ln\left(\frac{h}{r} + \sqrt{\left(\frac{h}{r}\right)^2 - 1}\right) \text{ H/m}$$

DIFFERENTIAL ELEMENTS

Cartesian coordinates

$$d\vec{l} = \hat{x}dx + \hat{y}dy + \hat{z}dz ; \quad d\vec{s} = \hat{x}dydz + \hat{y}dxdz + \hat{z}dxdy ; \quad dv = dxdydz$$

Cylindrical coordinates

$$d\vec{l} = \hat{r}dr + \hat{\varphi}rd\varphi + \hat{z}dz ; \quad d\vec{s} = \hat{r}rd\varphi dz + \hat{\varphi}drdz + \hat{z}rdrd\varphi ; \quad dv = rdrd\varphi dz$$

Spherical coordinates