CHAPTER 17

FOURIER TRANSFORM

No human investigation can claim to be scientific if it doesn't pass the test of mathematical proof.

—Leonardo da Vinci

Enhancing Your Career

Career in Communications Systems Communications systems apply the principles of circuit analysis. A communication system is designed to convey information from a source (the transmitter) to a destination (the receiver) via a channel (the propagation medium). Communications engineers design systems for transmitting and receiving information. The information can be in the form of voice, data, or video.

We live in the information age—news, weather, sports, shopping, financial, business inventory, and other sources make information available to us almost instantly via communications systems. Some obvious examples of communications systems are the telephone network, mobile cellular telephones, radio, cable TV, satellite TV, fax, and radar. Mobile radio, used by police and fire departments, aircraft, and various businesses is another example.

The field of communications is perhaps the fastest growing area in electrical engineering. The merging of the communications field with computer technology in recent years has led to digital data communications networks such as local area networks, metropolitan area networks, and broadband integrated services digital networks. For example, the Internet (the "information superhighway") allows educators, business people, and others to send electronic mail from their computers worldwide, log onto remote databases, and transfer files. The Internet has hit the world like a tidal wave and is drastically changing the way people do business, communicate, and get information. This trend will continue.

A communications systems engineer designs systems that provide high-quality information services. The systems include hardware for generating, transmitting, and receiving information signals. Communications engineers are employed in numerous communications industries and places where communications systems are routinely used. More and more government agencies, academic departments, and businesses are demanding faster and more accurate transmission of information. To meet these needs, communications engineers are in high demand. Therefore, the future is in communications and every electrical engineer must prepare accordingly.

Cordless phone. Source: M. Nemzow, Fast Ethernet Implementation and Migration Solutions [New York: McGraw-Hill, 1997], p. 176.

17.1 INTRODUCTION

Fourier series enable us to represent a periodic function as a sum of sinusoids and to obtain the frequency spectrum from the series. The Fourier transform allows us to extend the concept of a frequency spectrum to nonperiodic functions. The transform assumes that a nonperiodic function is a periodic function with an infinite period. Thus, the Fourier transform is an integral representation of a nonperiodic function that is analogous to a Fourier series representation of a periodic function.

The Fourier transform is an *integral transform* like the Laplace transform. It transforms a function in the time domain into the frequency domain. The Fourier transform is very useful in communications systems and digital signal processing, in situations where the Laplace transform does not apply. While the Laplace transform can only handle circuits with inputs for $t > 0$ with initial conditions, the Fourier transform can handle circuits with inputs for $t < 0$ as well as those for $t > 0$.

We begin by using a Fourier series as a stepping stone in defining the Fourier transform. Then we develop some of the properties of the Fourier transform. Next, we apply the Fourier transform in analyzing circuits. We discuss Parseval's theorem, compare the Laplace and Fourier transforms, and see how the Fourier transform is applied in amplitude modulation and sampling.

17.2 DEFINITION OF THE FOURIER TRANSFORM

We saw in the previous chapter that a nonsinusoidal periodic function can be represented by a Fourier series, provided that it satisfies the Dirichlet conditions. What happens if a function is not periodic? Unfortunately, there are many important nonperiodic functions—such as a unit step or an exponential function—that we cannot represent by a Fourier series. As we shall see, the Fourier transform allows a transformation from the time to the frequency domain, even if the function is not periodic.

Suppose we want to find the Fourier transform of a nonperiodic function $p(t)$, shown in Fig. 17.1(a). We consider a periodic function $f(t)$ whose shape over one period is the same as $p(t)$, as shown in Fig. 17.1(b). If we let the period $T \to \infty$, only a single pulse of width τ [the desired nonperiodic function in Fig. 17.1(a)] remains, because the adjacent pulses have been moved to infinity. Thus, the function $f(t)$ is no longer periodic. In other words, $f(t) = p(t)$ as $T \to \infty$. It is interesting to consider the spectrum of $f(t)$ for $A = 10$ and $\tau = 0.2$ (see Section 16.6). Figure 17.2 shows the effect of increasing T on the spectrum. First, we notice that the general shape of the spectrum remains the same, and the frequency at which the envelope first becomes zero remains the same. However, the amplitude of the spectrum and the spacing between adjacent components both decrease, while the number of harmonics increases. Thus, over a range of frequencies, the sum of the amplitudes of the harmonics remains almost constant. Since the total "strength" or energy of the components within a band must remain unchanged, the amplitudes of the harmonics must decrease as T increases. Since $f = 1/T$, as T increases, f or ω decreases, so that the discrete spectrum ultimately becomes continuous.

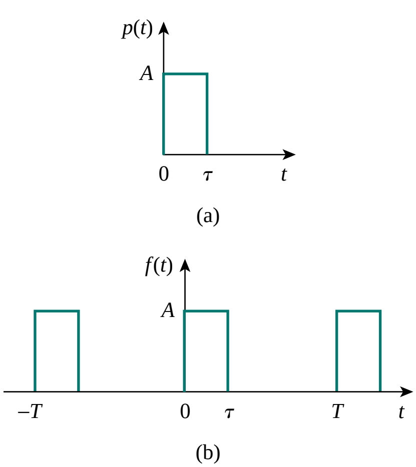

Figure 17.1 (a) A nonperiodic function, (b) increasing T to infinity makes $f(t)$ become the nonperiodic function in (a).

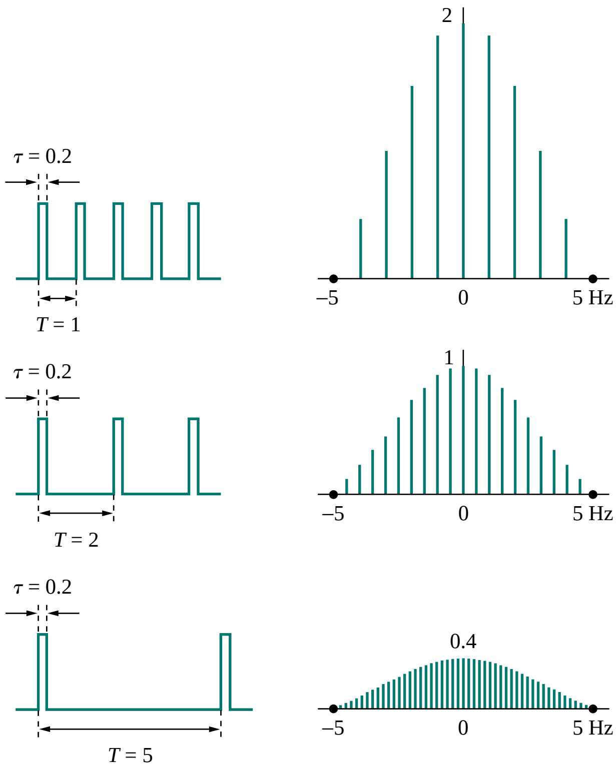

Figure 17.2 Effect of increasing T on the spectrum of the periodic pulse trains in Fig. 17.1(b).
(Source: L. Balmer, Signals and Systems: An Introduction [London: Prentice-Hall, 1991], p. 229.)

To further understand this connection between a nonperiodic function and its periodic counterpart, consider the exponential form of a Fourier series in Eq. (16.58), namely,

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \tag{17.1}$$

where

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}\,dt \tag{17.2}$$

The fundamental frequency is

$$\omega_0 = \frac{2\pi}{T} \tag{17.3}$$

and the spacing between adjacent harmonics is

$$\Delta\omega = (n+1)\omega_0 - n\omega_0 = \omega_0 = \frac{2\pi}{T} \tag{17.4}$$

Substituting Eq. (17.2) into Eq. (17.1) gives

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} \left[\frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t} \, dt \right] e^{jn\omega_0 t} \\ &= \sum_{n=-\infty}^{\infty} \left[\frac{\Delta\omega}{2\pi} \int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t} \, dt \right] e^{jn\omega_0 t} \\ &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left[\int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t} \, dt \right] \Delta\omega e^{jn\omega_0 t} \end{aligned} \tag{17.5}$$

If we let $T \to \infty$, the summation becomes integration, the incremental spacing $\Delta\omega$ becomes the differential separation $d\omega$, and the discrete harmonic frequency $n\omega_0$ becomes a continuous frequency ω. Thus, as $T \to \infty$,

$$\begin{aligned} \sum_{n=-\infty}^{\infty} &\implies \int_{-\infty}^{\infty} \\ \Delta\omega &\implies d\omega \\ n\omega_0 &\implies \omega \end{aligned} \tag{17.6}$$

so that Eq. (17.5) becomes

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(t) e^{-j\omega t} \, dt \right] e^{j\omega t} \, d\omega \tag{17.7}$$

Some authors use $F(j\omega)$ instead of $F(\omega)$ to represent the Fourier transform.

The term in the brackets is known as the *Fourier transform* of $f(t)$ and is represented by $F(\omega)$. Thus

$$F(\omega) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} \, dt \tag{17.8}$$

where $\mathcal{F}$ is the Fourier transform operator. It is evident from Eq. (17.8) that:

> The **Fourier transform** is an integral transformation of $f(t)$ from the time domain to the frequency domain.

In general, $F(\omega)$ is a complex function; its magnitude is called the *amplitude spectrum*, while its phase is called the *phase spectrum*. Thus $F(\omega)$ is the *spectrum*.

Equation (17.7) can be written in terms of $F(\omega)$, and we obtain the *inverse Fourier transform* as

$$f(t) = \mathcal{F}^{-1}[F(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} \, d\omega \tag{17.9}$$

The function $f(t)$ and its transform $F(\omega)$ form the Fourier transform pairs:

$$f(t) \iff F(\omega) \tag{17.10}$$

since one can be derived from the other.

The Fourier transform $F(\omega)$ exists when the Fourier integral in Eq. (17.8) converges. A sufficient but not necessary condition that $f(t)$ has a Fourier transform is that it be completely integrable in the sense that

$$\int_{-\infty}^{\infty} |f(t)|\, dt < \infty \tag{17.11}$$

For example, the Fourier transform of the unit ramp function $tu(t)$ does not exist, because the function does not satisfy the condition above.

To avoid the complex algebra that explicitly appears in the Fourier transform, it is sometimes expedient to temporarily replace $j\omega$ with s and then replace s with $j\omega$ at the end.

EXAMPLE 17.1

Find the Fourier transform of the following functions: (a) $\delta(t - t_0)$, (b) $e^{j\omega_0 t}$, (c) $\cos\omega_0 t$.

Solution:

(a) For the impulse function,

$$F(\omega) = \mathcal{F}[\delta(t - t_0)] = \int_{-\infty}^{\infty} \delta(t - t_0)e^{-j\omega t}\, dt = e^{-j\omega t_0} \tag{17.1.1}$$

where the sifting property of the impulse function in Eq. (7.32) has been applied. For the special case $t_0 = 0$, we obtain

$$\mathcal{F}[\delta(t)] = 1 \tag{17.1.2}$$

This shows that the magnitude of the spectrum of the impulse function is constant; that is, all frequencies are equally represented in the impulse function.

(b) We can find the Fourier transform of $e^{j\omega_0 t}$ in two ways. If we let

$$F(\omega) = \delta(\omega - \omega_0)$$

then we can find $f(t)$ using Eq. (17.9), writing

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0)e^{j\omega t}\, d\omega$$

Using the sifting property of the impulse function gives

$$f(t) = \frac{1}{2\pi}e^{j\omega_0 t}$$

Since $F(\omega)$ and $f(t)$ constitute a Fourier transform pair, so too must $2\pi\delta(\omega - \omega_0)$ and $e^{j\omega_0 t}$,

$$\mathcal{F}[e^{j\omega_0 t}] = 2\pi\delta(\omega - \omega_0) \tag{17.1.3}$$

Alternatively, from Eq. (17.1.2),

$$\delta(t) = \mathcal{F}^{-1}[1]$$

Using the inverse Fourier transform formula in Eq. (17.9),

$$\delta(t) = \mathcal{F}^{-1}[1] = \frac{1}{2\pi}\int_{-\infty}^{\infty} 1e^{j\omega t}\, d\omega$$

or

$$\int_{-\infty}^{\infty} e^{j\omega t}\, d\omega = 2\pi\,\delta(t) \tag{17.1.4}$$

Interchanging variables t and ω results in

$$\int_{-\infty}^{\infty} e^{j\omega t} dt = 2\pi\,\delta(\omega) \tag{17.1.5}$$

Using this result, the Fourier transform of the given function is

$$\mathcal{F}[e^{j\omega_0 t}] = \int_{-\infty}^{\infty} e^{j\omega_0 t} e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} e^{j(\omega_0-\omega)}\, dt = 2\pi\,\delta(\omega_0 - \omega)$$

Since the impulse function is an even function, with $\delta(\omega_0 - \omega) = \delta(\omega - \omega_0)$,

$$\mathcal{F}[e^{j\omega_0 t}] = 2\pi\,\delta(\omega - \omega_0) \tag{17.1.6}$$

By simply changing the sign of ω_0, we readily obtain

$$\mathcal{F}[e^{-j\omega_0 t}] = 2\pi\,\delta(\omega + \omega_0) \tag{17.1.7}$$

Also, by setting $\omega_0 = 0$,

$$\mathcal{F}[1] = 2\pi\,\delta(\omega) \tag{17.1.8}$$

(c) By using the result in Eqs. (17.1.6) and (17.1.7), we get

$$\begin{aligned}\mathcal{F}[\cos\omega_0 t] &= \mathcal{F}\left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right] \\ &= \frac{1}{2}\mathcal{F}[e^{j\omega_0 t}] + \frac{1}{2}\mathcal{F}[e^{-j\omega_0 t}] \\ &= \pi\,\delta(\omega - \omega_0) + \pi\,\delta(\omega + \omega_0)\end{aligned} \tag{17.1.9}$$

The Fourier transform of the cosine signal is shown in Fig. 17.3.

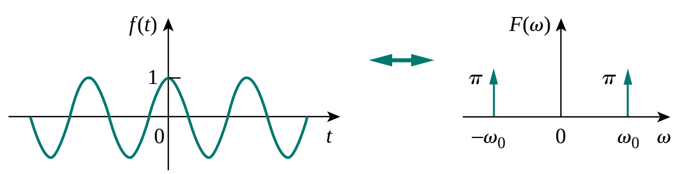

Figure 17.3 Fourier transform of $f(t) = \cos\omega_0 t$.

PRACTICE PROBLEM 17.1

Determine the Fourier transforms of the following functions: (a) gate function $g(t) = u(t-1) - u(t-2)$, (b) $4\delta(t+2)$, (c) $\sin\omega_0 t$.

Answer: (a) $(e^{-j\omega} - e^{-j2\omega})/j\omega$, (b) $4e^{j2\omega}$,
(c) $j\pi[\delta(\omega + \omega_0) - \pi\,\delta(\omega - \omega_0)]$.

EXAMPLE 17.2

Derive the Fourier transform of a single rectangular pulse of width τ and height A, shown in Fig. 17.4.

Solution:

$$F(\omega) = \int_{-\tau/2}^{\tau/2} Ae^{-j\omega t}\, dt = -\frac{A}{j\omega} e^{-j\omega t}\Big|_{-\tau/2}^{\tau/2}$$

$$= \frac{2A}{\omega}\left(\frac{e^{j\omega\tau/2} - e^{-j\omega\tau/2}}{2j}\right)$$

$$= A\tau \frac{\sin \omega\tau/2}{\omega\tau/2} = A\tau \text{ sinc } \frac{\omega\tau}{2}$$

If we make $A = 10$ and $\tau = 2$ as in Fig. 16.27 (like in Section 16.6), then

$$F(\omega) = 20 \text{ sinc } \omega$$

whose amplitude spectrum is shown in Fig. 17.5. Comparing Fig. 17.4 with the frequency spectrum of the rectangular pulses in Fig. 16.28, we notice that the spectrum in Fig. 16.28 is discrete and its envelope has the same shape as the Fourier transform of a single rectangular pulse.

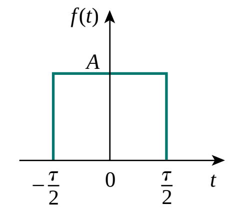

Figure 17.4 A rectangular pulse; for Example 17.2.

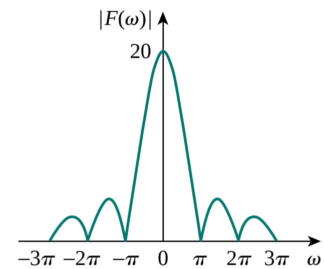

Figure 17.5 Amplitude spectrum of the rectangular pulse in Fig. 17.4; for Example 17.2.

PRACTICE PROBLEM 17.2

Obtain the Fourier transform of the function in Fig. 17.6.

Answer: $\dfrac{2(\cos\omega - 1)}{j\omega}$.

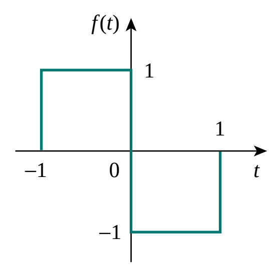

Figure 17.6 For Practice Prob. 17.2.

EXAMPLE 17.3

Obtain the Fourier transform of the "switched-on" exponential function shown in Fig. 17.7.

Solution:

From Fig. 17.7,

$$f(t) = e^{-at}u(t) = \begin{cases} e^{-at}, & t > 0 \\ 0, & t < 0 \end{cases}$$

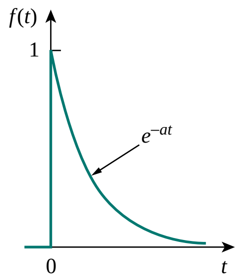

Figure 17.7 For Example 17.3.

Hence,

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt = \int_0^{\infty} e^{-at}e^{-j\omega t}\,dt = \int_0^{\infty} e^{-(a+j\omega)t}\,dt$$

$$= \frac{-1}{a+j\omega}e^{-(a+j\omega)t}\Big|_0^{\infty} = \frac{1}{a+j\omega}$$

PRACTICE PROBLEM 17.3

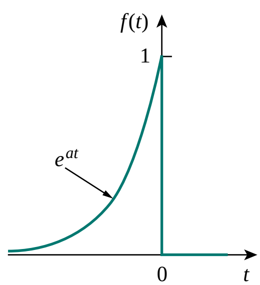

Figure 17.8 For Practice Prob. 17.3.

Determine the Fourier transform of the "switched-off" exponential function in Fig. 17.8.

Answer: $\dfrac{1}{a-j\omega}$.

17.3 PROPERTIES OF THE FOURIER TRANSFORM

We now develop some properties of the Fourier transform that are useful in finding the transforms of complicated functions from the transforms of simple functions. For each property, we will first state and derive it, and then illustrate it with some examples.

Linearity

If $F_1(\omega)$ and $F_2(\omega)$ are the Fourier transforms of $f_1(t)$ and $f_2(t)$, respectively, then

$$\mathcal{F}[a_1 f_1(t) + a_2 f_2(t)] = a_1 F_1(\omega) + a_2 F_2(\omega) \tag{17.12}$$

where a_1 and a_2 are constants. This property simply states that the Fourier transform of a linear combination of functions is the same as the linear combination of the transforms of the individual functions. The proof of the linearity property in Eq. (17.12) is straightforward. By definition,

$$\begin{aligned}\mathcal{F}[a_1 f_1(t) + a_2 f_2(t)] &= \int_{-\infty}^{\infty} [a_1 f_1(t) + a_2 f_2(t)]e^{-j\omega t}\,dt \\ &= \int_{-\infty}^{\infty} a_1 f_1(t)e^{-j\omega t}\,dt + \int_{-\infty}^{\infty} a_2 f_2(t)e^{-j\omega t}dt \\ &= a_1 F_1(\omega) + a_2 F_2(\omega)\end{aligned} \tag{17.13}$$

For example, $\sin\omega_0 t = \frac{1}{2j}(e^{j\omega_0 t} - e^{-j\omega_0 t})$. Using the linearity property,

$$F[\sin\omega_0 t] = \frac{1}{2j}[\mathcal{F}(e^{j\omega_0 t}) - \mathcal{F}(e^{-j\omega_0 t})] \\ = \frac{\pi}{j}[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)] \tag{17.14}$$

Time Scaling

If $F(\omega) = \mathcal{F}[f(t)]$, then

$$\mathcal{F}[f(at)] = \frac{1}{|a|}F\left(\frac{\omega}{a}\right) \tag{17.15}$$

where a is a constant. Equation (17.15) shows that time expansion ($|a| > 1$) corresponds to frequency compression, or conversely, time compression ($|a| < 1$) implies frequency expansion. The proof of the time-scaling property proceeds as follows.

$$\mathcal{F}[f(at)] = \int_{-\infty}^{\infty} f(at)e^{-j\omega t}\,dt \tag{17.16}$$

If we let $x = at$, so that $dx = a\,dt$, then

$$\mathcal{F}[f(at)] = \int_{-\infty}^{\infty} f(x)e^{-j\omega x/a}\,\frac{dx}{a} = \frac{1}{a}F\left(\frac{\omega}{a}\right) \tag{17.17}$$

For example, for the rectangular pulse $p(t)$ in Example 17.2,

$$\mathcal{F}[p(t)] = A\tau\ \text{sinc}\ \frac{\omega\tau}{2} \tag{17.18a}$$

Using Eq. (17.15),

$$\mathcal{F}[p(2t)] = \frac{A\tau}{2}\ \text{sinc}\ \frac{\omega\tau}{4} \tag{17.18b}$$

It may be helpful to plot $p(t)$ and $p(2t)$ and their Fourier transforms. Since

$$p(t) = \begin{cases} A, & -\dfrac{\tau}{2} < t < \dfrac{\tau}{2} \\ 0, & \text{otherwise} \end{cases} \tag{17.19a}$$

then replacing every t with $2t$ gives

$$p(2t) = \begin{cases} A, & -\dfrac{\tau}{2} < 2t < \dfrac{\tau}{2} \\ 0, & \text{otherwise} \end{cases} = \begin{cases} A, & -\dfrac{\tau}{4} < t < \dfrac{\tau}{4} \\ 0, & \text{otherwise} \end{cases} \tag{17.19b}$$

showing that $p(2t)$ is time compressed, as shown in Fig. 17.9(b). To plot both Fourier transforms in Eq. (17.18), we recall that the sinc function has zeros when its argument is $n\pi$, where n is an integer. Hence, for the transform of $p(t)$ in Eq. (17.18a), $\omega\tau/2 = 2\pi f\tau/2 = n\pi \to f = n/\tau$, and for the transform of $p(2t)$ in Eq. (17.18b), $\omega\tau/4 = 2\pi f\tau/4 = n\pi \to f = 2n/\tau$. The plots of the Fourier transforms are shown in Fig. 17.9, which shows that time compression corresponds with frequency expansion. We should expect this intuitively, because when the signal is squashed in time, we expect it to change more rapidly, thereby causing higher-frequency components to exist.

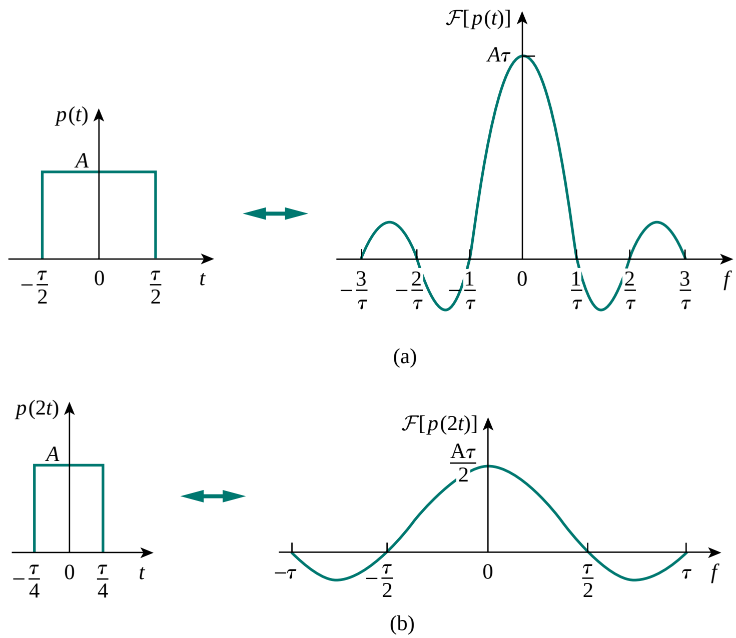

Figure 17.9 The effect of time scaling: (a) transform of the pulse, (b) time compression of the pulse causes frequency expansion.

Time Shifting

If $F(\omega) = \mathcal{F}[f(t)]$, then

$$\mathcal{F}[f(t - t_0)] = e^{-j\omega t_0} F(\omega) \tag{17.20}$$

that is, a delay in the time domain corresponds to a phase shift in the frequency domain. To derive the time shifting property, we note that

$$\mathcal{F}[f(t - t_0)] = \int_{-\infty}^{\infty} f(t - t_0) e^{-j\omega t}\, dt \tag{17.21}$$

If we let $x = t - t_0$ so that $dx = dt$ and $t = x + t_0$, then

$$\begin{aligned}\mathcal{F}[f(t - t_0)] &= \int_{-\infty}^{\infty} f(x) e^{-j\omega(x+t_0)}\, dx \\ &= e^{-j\omega t_0} \int_{-\infty}^{\infty} f(x) e^{-j\omega x}\, dx = e^{-j\omega t_0} F(\omega)\end{aligned} \tag{17.22}$$

Similarly, $\mathcal{F}[f(t + t_0)] = e^{j\omega t_0} F(\omega)$.

For example, from Example 17.3,

$$\mathcal{F}[e^{-at} u(t)] = \frac{1}{a + j\omega} \tag{17.23}$$

The transform of $f(t) = e^{-(t-2)}u(t-2)$ is

$$F(\omega) = \mathcal{F}[e^{-(t-2)}u(t-2)] = \frac{e^{-j2\omega}}{1+j\omega} \tag{17.24}$$

Frequency Shifting (or Amplitude Modulation)

This property states that if $F(\omega) = \mathcal{F}[f(t)]$, then

$$\boxed{\mathcal{F}[f(t)e^{j\omega_0 t}] = F(\omega-\omega_0)} \tag{17.25}$$

meaning, a frequency shift in the frequency domain adds a phase shift to the time function. By definition,

$$\begin{aligned}\mathcal{F}[f(t)e^{j\omega_0 t}] &= \int_{-\infty}^{\infty} f(t)e^{j\omega_0 t}e^{-j\omega t}\,dt \\ &= \int_{-\infty}^{\infty} f(t)e^{-j(\omega-\omega_0)t}\,dt = F(\omega-\omega_0)\end{aligned} \tag{17.26}$$

For example, $\cos\omega_0 t = \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})$. Using the property in Eq. (17.25),

$$\begin{aligned}\mathcal{F}[f(t)\cos\omega_0 t] &= \frac{1}{2}\mathcal{F}\left[f(t)e^{j\omega_0 t}\right] + \frac{1}{2}\mathcal{F}\left[f(t)e^{-j\omega_0 t}\right] \\ &= \frac{1}{2}F(\omega-\omega_0) + \frac{1}{2}F(\omega+\omega_0)\end{aligned} \tag{17.27}$$

This is an important result in modulation where frequency components of a signal are shifted. If, for example, the amplitude spectrum of $f(t)$ is as shown in Fig. 17.10(a), then the amplitude spectrum of $f(t)\cos\omega_0 t$ will be as shown in Fig. 17.10(b). We will elaborate on amplitude modulation in Section 17.7.1.

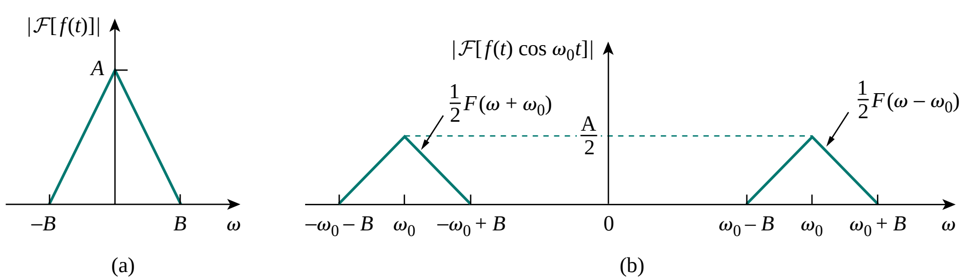

Figure 17.10 Amplitude spectra of: (a) signal $f(t)$, (b) modulated signal $f(t)\cos\omega t$.

Time Differentiation

Given that $F(\omega) = \mathcal{F}[f(t)]$, then

$$\boxed{\mathcal{F}[f'(t)] = j\omega F(\omega)} \tag{17.28}$$

In other words, the transform of the derivative of $f(t)$ is obtained by multiplying the transform of $f(t)$ by $j\omega$. By definition,

$$f(t) = \mathcal{F}^{-1}[F(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega \tag{17.29}$$

Taking the derivative of both sides with respect to t gives

$$f'(t) = \frac{j\omega}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega = j\omega\mathcal{F}^{-1}[F(\omega)]$$

or

$$\mathcal{F}[f'(t)] = j\omega F(\omega) \tag{17.30}$$

Repeated applications of Eq. (17.30) give

$$\boxed{\mathcal{F}[f^{(n)}(t)] = (j\omega)^n F(\omega)} \tag{17.31}$$

For example, if $f(t) = e^{-at}$, then

$$f'(t) = -ae^{-at} = -af(t) \tag{17.32}$$

Taking the Fourier transforms of the first and last terms, we obtain

$$j\omega F(\omega) = -aF(\omega) \qquad \Longrightarrow \qquad F(\omega) = \frac{1}{a + j\omega} \tag{17.33}$$

which agrees with the result in Example 17.3.

Time Integration

Given that $F(\omega) = \mathcal{F}[f(t)]$, then

$$\boxed{\mathcal{F}\left[\int_{-\infty}^{t} f(t)\,dt\right] = \frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega)} \tag{17.34}$$

that is, the transform of the integral of $f(t)$ is obtained by dividing the transform of $f(t)$ by $j\omega$ and adding the result to the impulse term that reflects the dc component $F(0)$. Someone might ask, "How do we know that when we take the Fourier transform for time integration, we should integrate over the interval $[-\infty, t]$ and not $[-\infty, \infty]$?" When we integrate over $[-\infty, \infty]$, the result does not depend on time anymore, and the Fourier transform of a constant is what we will eventually get. But when we integrate over $[-\infty, t]$, we get the integral of the function from the past to time t, so that the result depends on t and we can take the Fourier transform of that.

If ω is replaced by 0 in Eq. (17.8),

$$F(0) = \int_{-\infty}^{\infty} f(t)\,dt \tag{17.35}$$

indicating that the dc component is zero when the integral of $f(t)$ over all time vanishes. The proof of the time integration in Eq. (17.34) will be given later when we consider the convolution property.

For example, we know that $\mathcal{F}[\delta(t)] = 1$ and that integrating the impulse function gives the unit step function [see Eq. (7.39a)]. By applying the property in Eq. (17.34), we obtain the Fourier transform of the unit step function as

$$\mathcal{F}[u(t)] = \mathcal{F}\left[\int_{-\infty}^{t} \delta(t)\,dt\right] = \frac{1}{j\omega} + \pi\delta(\omega) \tag{17.36}$$

Reversal

If $F(\omega) = \mathcal{F}[f(t)]$, then

$$\mathcal{F}[f(-t)] = F(-\omega) = F^*(\omega) \tag{17.37}$$

where the asterisk denotes the complex conjugate. This property states that reversing $f(t)$ about the time axis reverses $F(\omega)$ about the frequency axis. This may be regarded as a special case of time scaling for which $a = -1$ in Eq. (17.15).

Duality

This property states that if $F(\omega)$ is the Fourier transform of $f(t)$, then the Fourier transform of $F(t)$ is $2\pi f(-\omega)$; we write

$$\mathcal{F}[f(t)] = F(\omega) \quad \Longrightarrow \quad \mathcal{F}[F(t)] = 2\pi f(-\omega) \tag{17.38}$$

This expresses the symmetry property of the Fourier transform. To derive this property, we recall that

$$f(t) = \mathcal{F}^{-1}[F(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\, d\omega$$

or

$$2\pi f(t) = \int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\, d\omega \tag{17.39}$$

Replacing t by $-t$ gives

$$2\pi f(-t) = \int_{-\infty}^{\infty} F(\omega)e^{-j\omega t}\, d\omega$$

If we interchange t and ω, we obtain

$$2\pi f(-\omega) = \int_{-\infty}^{\infty} F(t)e^{-j\omega t}\, dt = \mathcal{F}[F(t)] \tag{17.40}$$

as expected.

For example, if $f(t) = e^{-|t|}$, then

$$F(\omega) = \frac{2}{\omega^2 + 1} \tag{17.41}$$

Since $f(t)$ is the sum of the signals in Figs. 17.7 and 17.8, $F(\omega)$ is the sum of the results in Example 17.3 and Practice Prob. 17.3.

By the duality property, the Fourier transform of $F(t) = 2/(t^2 + 1)$ is

$$2\pi f(\omega) = 2\pi e^{-|\omega|} \tag{17.42}$$

Figure 17.11 shows another example of the duality property. It illustrates the fact that if $f(t) = \delta(t)$ so that $F(\omega) = 1$, as in Fig. 17.11(a), then the Fourier transform of $F(t) = 1$ is $2\pi f(\omega) = 2\pi\delta(\omega)$ as shown in Fig. 17.11(b).

Convolution

Recall from Chapter 15 that if $x(t)$ is the input excitation to a circuit with an impulse function of $h(t)$, then the output response $y(t)$ is given by the convolution integral

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\lambda)x(t - \lambda)\, d\lambda \tag{17.43}$$

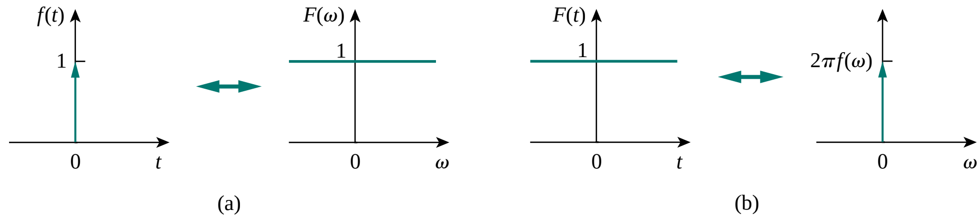

Figure 17.11 A typical illustration of the duality property of the Fourier transform: (a) transform of impulse, (b) transform of unit dc level.

If $X(\omega)$, $H(\omega)$, and $Y(\omega)$ are the Fourier transforms of $x(t)$, $h(t)$, and $y(t)$, respectively, then

$$Y(\omega) = \mathcal{F}[h(t) * x(t)] = H(\omega)X(\omega) \tag{17.44}$$

which indicates that convolution in the time domain corresponds with multiplication in the frequency domain.

To derive the convolution property, we take the Fourier transform of both sides of Eq. (17.43) to get

$$Y(\omega) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)\, d\lambda \right] e^{-j\omega t}\, dt \tag{17.45}$$

Exchanging the order of integration and factoring $h(\lambda)$, which does not depend on t, we have

$$Y(\omega) = \int_{-\infty}^{\infty} h(\lambda) \left[\int_{-\infty}^{\infty} x(t-\lambda)e^{-j\omega t}\, dt \right] d\lambda$$

For the integral within the brackets, let $\tau = t - \lambda$ so that $t = \tau + \lambda$ and $dt = d\tau$. Then,

$$\begin{aligned} Y(\omega) &= \int_{-\infty}^{\infty} h(\lambda) \left[\int_{-\infty}^{\infty} x(\tau)e^{-j\omega(\tau+\lambda)}\, d\tau \right] d\lambda \\ &= \int_{-\infty}^{\infty} h(\lambda)e^{-j\omega\lambda}\, d\lambda \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}\, d\tau = H(\omega)X(\omega) \end{aligned} \tag{17.46}$$

as expected. This result expands the phasor method beyond what was done with the Fourier series in the previous chapter.

The important relationship in Eq. (17.46) is the key reason for using the Fourier transform in the analysis of linear systems.

To illustrate the convolution property, suppose both $h(t)$ and $x(t)$ are identical rectangular pulses, as shown in Fig. 17.12(a) and 17.12(b). We recall from Example 17.2 and Fig. 17.5 that the Fourier transforms of the rectangular pulses are sinc functions, as shown in Fig. 17.12(c) and 17.12(d). According to the convolution property, the product of the sinc functions should give us the convolution of the rectangular pulses in the time domain. Thus, the convolution of the pulses in Fig. 17.12(e) and the product of the sinc functions in Fig. 17.12(f) form a Fourier pair.

In view of the duality property, we expect that if convolution in the time domain corresponds with multiplication in the frequency domain, then multiplication in the time domain should have a correspondence in the frequency domain. This happens to be the case. If $f(t) = f_1(t)f_2(t)$,

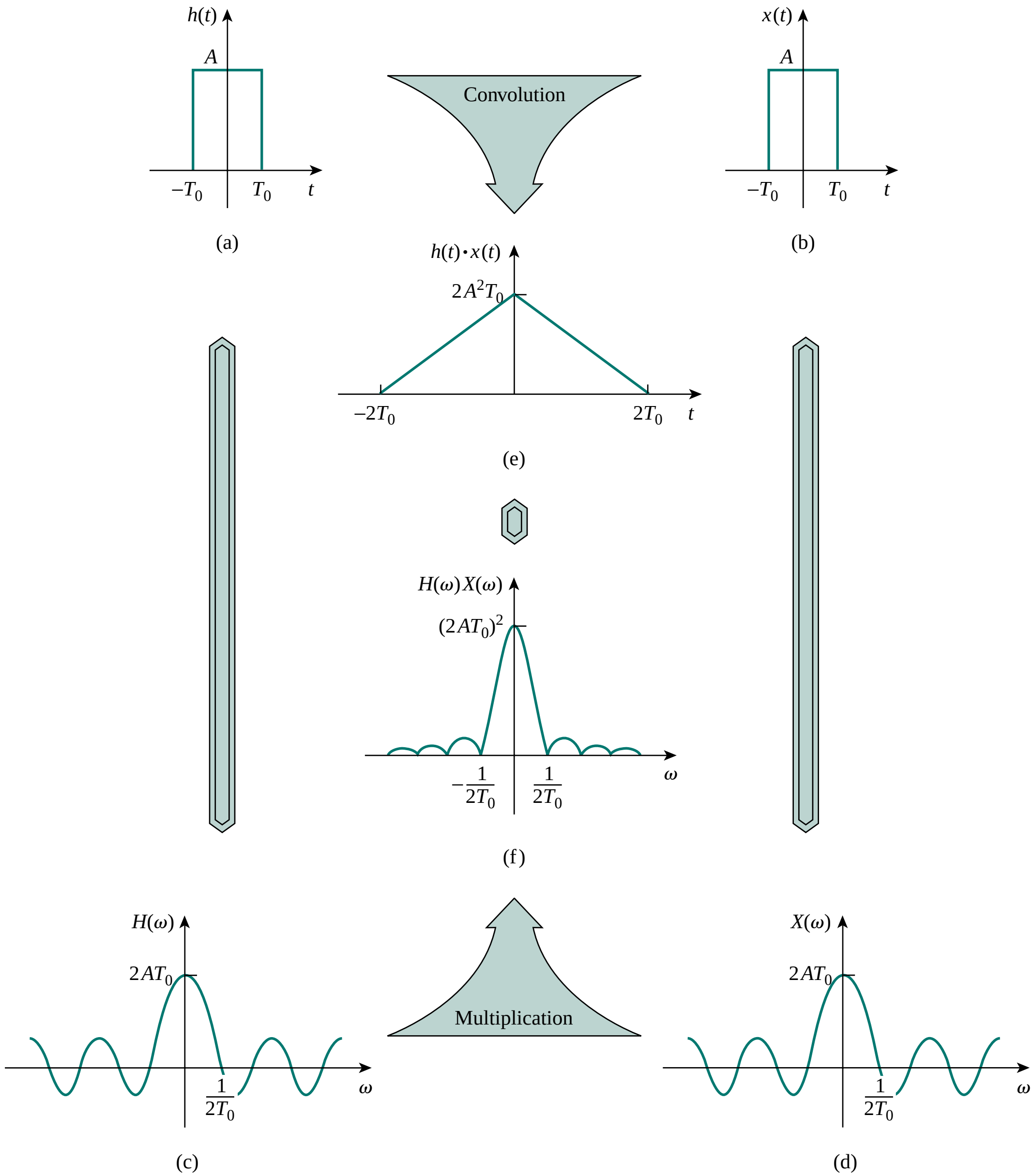

Figure 17.12 Graphical illustration of the convolution property.
(Source: E. O. Brigham, The Fast Fourier Transform [Englewood Cliffs, NJ: Prentice Hall, 1974], p. 60.)

then

$$F(\omega) = \mathcal{F}[f_1(t) f_2(t)] = \frac{1}{2\pi} F_1(\omega) * F_2(\omega) \tag{17.47}$$

or

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_1(\lambda) F_2(\omega - \lambda)\, d\lambda \tag{17.48}$$

which is convolution in the frequency domain. The proof of Eq. (17.48) readily follows from the duality property in Eq. (17.38).

Let us now derive the time integration property in Eq. (17.34). If we replace $x(t)$ with the unit step function $u(t)$ and $h(t)$ with $f(t)$ in Eq. (17.43), then

$$\int_{-\infty}^{\infty} f(\lambda)u(t-\lambda)\,d\lambda = f(t) * u(t) \tag{17.49}$$

But by the definition of the unit step function,

$$u(t-\lambda) = \begin{cases} 1, & t-\lambda > 0 \\ 0, & t-\lambda > 0 \end{cases}$$

We can write this as

$$u(t-\lambda) = \begin{cases} 1, & \lambda < t \\ 0, & \lambda > t \end{cases}$$

Substituting this into Eq. (17.49) makes the interval of integration change from $[-\infty, \infty]$ to $[-\infty, t]$, and thus Eq. (17.49) becomes

$$\int_{-\infty}^{t} f(\lambda)\,d\lambda = u(t) * f(t)$$

Taking the Fourier transform of both sides yields

$$\mathcal{F}\left[\int_{-\infty}^{t} f(\lambda)\,d\lambda\right] = U(\omega)F(\omega) \tag{17.50}$$

But from Eq. (17.36), the Fourier transform of the unit step function is

$$U(\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$$

Substituting this into Eq. (17.50) gives

$$\begin{aligned} \mathcal{F}\left[\int_{-\infty}^{t} f(\lambda)\,d\lambda\right] &= \left(\frac{1}{j\omega} + \pi\delta(\omega)\right)F(\omega) \\ &= \frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega) \end{aligned} \tag{17.51}$$

which is the time integration property of Eq. (17.34). Note that in Eq. (17.51), $F(\omega)\delta(\omega) = F(0)\delta(\omega)$, since $\delta(\omega)$ is only nonzero at $\omega = 0$.

Table 17.1 lists these properties of the Fourier transform. Table 17.2 presents the transform pairs of some common functions. Note the similarities between these tables and Tables 15.1 and 15.2.

TABLE 17.1 Properties of the Fourier transform.

Property	$f(t)$	$F(\omega)$
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(\omega) + a_2 F_2(\omega)$
Scaling	$f(at)$	$\frac{1}{\lvert a\rvert} F\left(\frac{\omega}{a}\right)$
Time shift	$f(t-a)u(t-a)$	$e^{-j\omega a}F(\omega)$
Frequency shift	$e^{j\omega_0 t} f(t)$	$F(\omega-\omega_0)$
Modulation	$\cos(\omega_0 t)\ f(t)$	$\frac{1}{2}[F(\omega+\omega_0) + F(\omega-\omega_0)]$

TABLE 17.1 *(continued)*

Property	$f(t)$	$F(\omega)$
Time differentiation	$\frac{df}{dt}$	$j\omega F(\omega)$
	$\frac{d^n f}{dt^n}$	$(j\omega)^n F(\omega)$
Time integration	$\int_{-\infty}^{t} f(t)\,dt$	$\frac{F(\omega)}{j\omega} + \pi F(0)\,\delta(\omega)$
Frequency differentiation	$t^n f(t)$	$(j)^n \frac{d^n}{d\omega^n} F(\omega)$
Reversal	$f(-t)$	$F(-\omega)$ or $F^*(\omega)$
Duality	$F(t)$	$2\pi f(-\omega)$
Convolution in t	$f_1(t) * f_1(t)$	$F_1(\omega) F_2(\omega)$
Convolution in ω	$f_1(t) f_1(t)$	$\frac{1}{2\pi} F_1(\omega) * F_2(\omega)$

TABLE 17.2 Fourier transform pairs.

$f(t)$	$F(\omega)$
$\delta(t)$	1
1	$2\pi\,\delta(\omega)$
$u(t)$	$\pi\,\delta(\omega) + \frac{1}{j\omega}$
$u(t+\tau) - u(t-\tau)$	$2\frac{\sin \omega\tau}{\omega}$
$\|t\|$	$\frac{-2}{\omega^2}$
$\text{sgn}(t)$	$\frac{2}{j\omega}$
$e^{-at}u(t)$	$\frac{1}{a + j\omega}$
$e^{at}u(-t)$	$\frac{1}{a - j\omega}$
$t^n e^{-at} u(t)$	$\frac{n!}{(a + j\omega)^{n+1}}$
$e^{-a\|t\|}$	$\frac{2a}{a^2 + \omega^2}$
$e^{j\omega_0 t}$	$2\pi\,\delta(\omega - \omega_0)$
$\sin \omega_0 t$	$j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$
$\cos \omega_0 t$	$\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$
$e^{-at} \sin \omega_0 t\, u(t)$	$\frac{\omega_0}{(a + j\omega)^2 + \omega_0^2}$
$e^{-at} \cos \omega_0 t\, u(t)$	$\frac{a + j\omega}{(a + j\omega)^2 + \omega_0^2}$

EXAMPLE 17.4

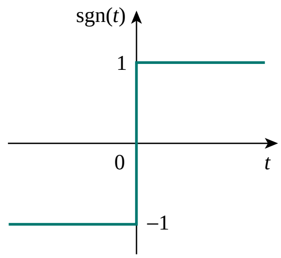

Figure 17.13 The signum function of Example 17.4.

Find the Fourier transforms of the following functions: (a) signum function sgn(t), shown in Fig. 17.13, (b) the double-sided exponential $e^{-a|t|}$, and (c) the sinc function $(\sin t)/t$.

Solution:

(a) We can obtain the Fourier transform of the *signum* function in three ways. First, we can write the signum function in terms of the unit step function as

$$\text{sgn}(t) = f(t) = u(t) - u(-t)$$

But from Eq. (17.36),

$$U(\omega) = \mathcal{F}[u(t)] = \pi\delta(\omega) + \frac{1}{j\omega}$$

Applying this and the reversal property, we obtain

$$\begin{aligned}\mathcal{F}[\text{sgn}(t)] &= U(\omega) - U(-\omega)\\ &= \left(\pi\delta(\omega) + \frac{1}{j\omega}\right) - \left(\pi\delta(-\omega) + \frac{1}{-j\omega}\right) = \frac{2}{j\omega}\end{aligned}$$

Second, another way of writing the signum function in terms of the unit step function is

$$f(t) = \text{sgn}(t) = -1 + 2u(t)$$

Taking the Fourier transform of each term gives

$$F(\omega) = -2\pi\delta(\omega) + 2\left(\pi\delta(\omega) + \frac{1}{j\omega}\right) = \frac{2}{j\omega}$$

Third, we can take the derivative of the signum function in Fig. 17.13 and obtain

$$f'(t) = 2\delta(t)$$

Taking the transform of this,

$$j\omega F(\omega) = 2 \qquad \Longrightarrow \qquad F(\omega) = \frac{2}{j\omega}$$

as obtained previously.

(b) The double-sided exponential can be expressed as

$$f(t) = e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t) = y(t) + y(-t)$$

where $y(t) = e^{-at}u(t)$ so that $Y(\omega) = 1/(a + j\omega)$. Applying the reversal property,

$$\mathcal{F}[e^{-a|t|}] = Y(\omega) + Y(-\omega) = \left(\frac{1}{a + j\omega} + \frac{1}{a - j\omega}\right) = \frac{2a}{a^2 + \omega^2}$$

(c) From Example 17.2,

$$\mathcal{F}\left[u\left(t + \frac{\tau}{2}\right) - u\left(t - \frac{\tau}{2}\right)\right] = \tau\frac{\sin(\omega\tau/2)}{\omega\tau/2} = \tau \text{ sinc } \frac{\omega\tau}{2}$$

Setting $\tau/2 = 1$ gives

$$\mathcal{F}[u(t+1) - u(t-1)] = 2\frac{\sin\omega}{\omega}$$

Applying the duality property,

$$\mathcal{F}\left[2\frac{\sin t}{t}\right] = 2\pi[U(\omega+1) - U(\omega-1)]$$

or

$$\mathcal{F}\left[\frac{\sin t}{t}\right] = \pi[U(\omega+1) - U(\omega-1)]$$

PRACTICE PROBLEM 17.4

Determine the Fourier transforms of these functions: (a) gate function $g(t) = u(t) - u(t-1)$, (b) $f(t) = te^{-2t}u(t)$, and (c) sawtooth pulse $f(t) = 10t[u(t) - u(t-2)]$.

Answer: (a) $(1 - e^{-j\omega})\left[\pi\delta(\omega) + \frac{1}{j\omega}\right]$, (b) $\frac{1}{(2+j\omega)^2}$, (c) $\frac{10(e^{-j2\omega} - 1)}{\omega^2} + \frac{20j}{\omega}e^{-j2\omega}$.

EXAMPLE 17.5

Find the Fourier transform of the function in Fig. 17.14.

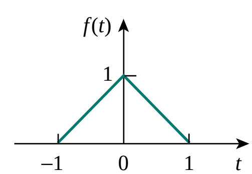

Figure 17.14 For Example 17.5.

Solution:

The Fourier transform can be found directly using Eq. (17.8), but it is much easier to find it using the derivative property. We can express the function as

$$f(t) = \begin{cases} 1+t, & -1 < t < 0 \\ 1-t, & 0 < t < 1 \end{cases}$$

Its first derivative is shown in Fig. 17.15(a) and is given by

$$f'(t) = \begin{cases} 1, & -1 < t < 0 \\ -1, & 0 < t < 1 \end{cases}$$

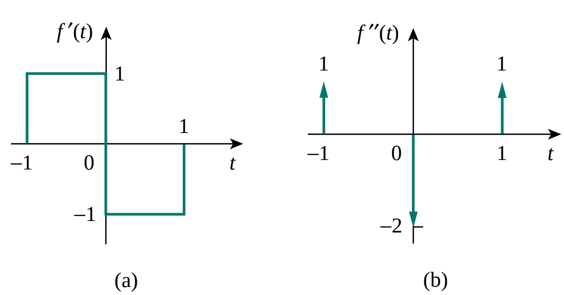

Figure 17.15 First and second derivatives of $f(t)$ in Fig. 17.14; for Example 17.5.

Its second derivative is in Fig. 17.15(b) and is given by

$$f''(t) = \delta(t+1) - 2\delta(t) + \delta(t-1)$$

Taking the Fourier transform of both sides,

$$(j\omega)^2 F(\omega) = e^{j\omega} - 2 + e^{-j\omega} = -2 + 2\cos\omega$$

or

$$F(\omega) = \frac{2(1-\cos\omega)}{\omega^2}$$

PRACTICE PROBLEM 17.5

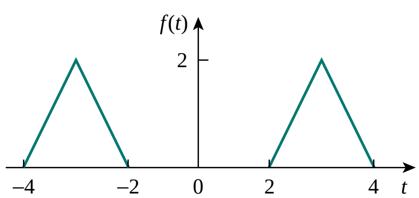

Figure 17.16 For Practice Prob. 17.5.

Determine the Fourier transform of the function in Fig. 17.16.

Answer: $(8\cos 3\omega - 4\cos 4\omega - 4\cos 2\omega)/\omega^2$.

EXAMPLE 17.6

Obtain the inverse Fourier transform of:

(a) $F(\omega) = \dfrac{10j\omega + 4}{(j\omega)^2 + 6j\omega + 8}$ (b) $G(\omega) = \dfrac{\omega^2 + 21}{\omega^2 + 9}$

Solution:

(a) To avoid complex algebra, we can replace $j\omega$ with s for the moment. Using partial fraction expansion,

$$F(s) = \frac{10s+4}{s^2+6s+8} = \frac{10s+4}{(s+4)(s+2)} = \frac{A}{s+4} + \frac{B}{s+2}$$

where

$$A = (s+4)F(s)\Big|_{s=-4} = \frac{10s+4}{(s+2)}\Big|_{s=-4} = \frac{-36}{-2} = 18$$

$$B = (s+2)F(s)\Big|_{s=-2} = \frac{10s+4}{(s+4)}\Big|_{s=-2} = \frac{-16}{2} = -8$$

Substituting $A = 18$ and $B = -8$ in $F(s)$ and s with $j\omega$ gives

$$F(j\omega) = \frac{18}{j\omega+4} + \frac{-8}{j\omega+2}$$

With the aid of Table 17.2, we obtain the inverse transform as

$$f(t) = (18e^{-4t} - 8e^{-2t})u(t)$$

(b) We simplify $G(\omega)$ as

$$G(\omega) = \frac{\omega^2+21}{\omega^2+9} = 1 + \frac{12}{\omega^2+9}$$

With the aid of Table 17.2, the inverse transform is obtained as

$$g(t) = \delta(t) + 2e^{-3|t|}$$

PRACTICE PROBLEM 17.6

Find the inverse Fourier transform of:

(a) $H(\omega) = \dfrac{6(3 + j2\omega)}{(1 + j\omega)(4 + j\omega)(2 + j\omega)}$

(b) $Y(\omega) = \pi\delta(\omega) + \dfrac{1}{j\omega} + \dfrac{2(1 + j\omega)}{(1 + j\omega)^2 + 16}$

Answer: (a) $h(t) = (2e^{-t} + 3e^{-2t} - 5e^{-4t})u(t)$,
(b) $y(t) = (1 + 2e^{-t}\cos 4t)u(t)$.

17.4 CIRCUIT APPLICATIONS

The Fourier transform generalizes the phasor technique to nonperiodic functions. Therefore, we apply Fourier transforms to circuits with nonsinusoidal excitations in exactly the same way we apply phasor techniques to circuits with sinusoidal excitations. Thus, Ohm's law is still valid:

$$V(\omega) = Z(\omega)I(\omega) \tag{17.52}$$

where $V(\omega)$ and $I(\omega)$ are the Fourier transforms of the voltage and current and $Z(\omega)$ is the impedance. We get the same expressions for the impedances of resistors, inductors, and capacitors as in phasor analysis, namely,

$$\begin{array}{lcl} R & \Longrightarrow & R \\ L & \Longrightarrow & j\omega L \\ C & \Longrightarrow & \dfrac{1}{j\omega C} \end{array} \tag{17.53}$$

Once we transform the functions for the circuit elements into the frequency domain and take the Fourier transforms of the excitations, we can use circuit techniques such as voltage division, source transformation, mesh analysis, node analysis, or Thevenin's theorem, to find the unknown response (current or voltage). Finally, we take the inverse Fourier transform to obtain the response in the time domain.

Although the Fourier transform method produces a response that exists for $-\infty < t < \infty$, Fourier analysis cannot handle circuits with initial conditions.

The transfer function is again defined as the ratio of the output response $Y(\omega)$ to the input excitation $X(\omega)$, that is,

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} \tag{17.54}$$

or

$$Y(\omega) = H(\omega)X(\omega) \tag{17.55}$$

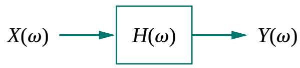

Figure 17.17 Input-output relationship of a circuit in the frequency-domain.

The frequency-domain input-output relationship is portrayed in Fig. 17.17. Equation (17.55) shows that if we know the transfer function and the input, we can readily find the output. The relationship in Eq. (17.54) is the principal reason for using the Fourier transform in circuit analysis. Notice that $H(\omega)$ is identical to $H(s)$ with $s = j\omega$. Also, if the input is an impulse function [i.e., $x(t) = \delta(t)$], then $X(\omega) = 1$, so that the response is

$$Y(\omega) = H(\omega) = \mathcal{F}[h(t)] \tag{17.56}$$

indicating that $H(\omega)$ is the Fourier transform of the impulse response $h(t)$.

EXAMPLE 17.7

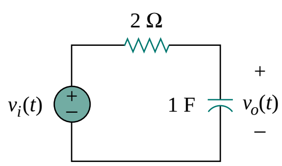

Figure 17.18 For Example 17.7.

Find $v_o(t)$ in the circuit of Fig. 17.18 for $v_i(t) = 2e^{-3t}u(t)$.

Solution:

The Fourier transform of the input voltage is

$$V_i(\omega) = \frac{2}{3 + j\omega}$$

and the transfer function obtained by voltage division is

$$H(\omega) = \frac{V_o(\omega)}{V_i(\omega)} = \frac{1/j\omega}{2 + 1/j\omega} = \frac{1}{1 + j2\omega}$$

Hence,

$$V_o(\omega) = V_i(\omega)H(\omega) = \frac{2}{(3 + j\omega)(1 + j2\omega)}$$

or

$$V_o(\omega) = \frac{1}{(3 + j\omega)(0.5 + j\omega)}$$

By partial fractions,

$$V_o(\omega) = \frac{-0.4}{3 + j\omega} + \frac{0.4}{0.5 + j\omega}$$

Taking the inverse Fourier transform yields

$$v_o(t) = 0.4(e^{-0.5t} - e^{-3t})u(t)$$

PRACTICE PROBLEM 17.7

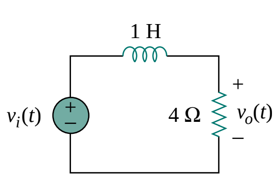

Figure 17.19 For Practice Prob. 17.7.

Determine $v_o(t)$ in Fig. 17.19 if $v_i(t) = 2\,\mathrm{sgn}(t) = -2 + 4u(t)$.

Answer: $-2 + 4(1 - e^{-4t})u(t)$.

EXAMPLE 17.8

Using the Fourier transform method, find $i_o(t)$ in Fig. 17.20 when $i_s(t) = 10 \sin 2t$ A.

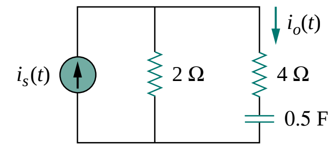

Figure 17.20 For Example 17.8.

Solution:

By current division,

$$H(\omega) = \frac{I_o(\omega)}{I_s(\omega)} = \frac{2}{2+4+2/j\omega} = \frac{j\omega}{1+j\omega 3}$$

If $i_s(t) = 10 \sin 2t$, then

$$I_s(\omega) = j\pi 10[\delta(\omega+2) - \delta(\omega-2)]$$

Hence,

$$I_o(\omega) = H(\omega) I_s(\omega) = \frac{10\pi\omega[\delta(\omega-2) - \delta(\omega+2)]}{1+j\omega 3}$$

The inverse Fourier transform of $I_o(\omega)$ cannot be found using Table 17.2. We resort to the inverse Fourier transform formula in Eq. (17.9) and write

$$i_o(t) = \mathcal{F}^{-1}[I_o(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{10\pi\omega[\delta(\omega-2) - \delta(\omega+2)]}{1+j\omega 3} e^{j\omega t}\, d\omega$$

We apply the sifting property of the impulse function, namely,

$$\delta(\omega-\omega_0) f(\omega) = f(\omega_0)$$

or

$$\int_{-\infty}^{\infty} \delta(\omega-\omega_0) f(\omega)\, d\omega = f(\omega_0)$$

and obtain

$$\begin{aligned} i_o(t) &= \frac{10\pi}{2\pi}\left[\frac{2}{1+j6}e^{j2t} - \frac{-2}{1-j6}e^{-j2t}\right] \\ &= 10\left[\frac{e^{j2t}}{6.082e^{j80.54^\circ}} + \frac{e^{-j2t}}{6.082e^{-j80.54^\circ}}\right] \\ &= 1.644[e^{j(2t-80.54^\circ)} + e^{-j(2t-80.54^\circ)}] \\ &= 3.288\cos(2t - 80.54^\circ) \text{ A} \end{aligned}$$

PRACTICE PROBLEM 17.8

Find the current $i_o(t)$ in the circuit in Fig. 17.21, given that $i_s(t) = 20 \cos 4t$ A.

Answer: $11.8\cos(4t + 26.57^\circ)$ A.

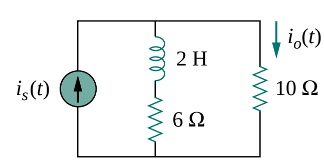

Figure 17.21 For Practice Prob. 17.8.

17.5 PARSEVAL'S THEOREM

Parseval's theorem demonstrates one practical use of the Fourier transform. It relates the energy carried by a signal to the Fourier transform of the signal. If $p(t)$ is the power associated with the signal, the energy carried by the signal is

$$W = \int_{-\infty}^{\infty} p(t)\, dt \tag{17.57}$$

In order to be able compare the energy content of current and voltage signals, it is convenient to use a 1-Ω resistor as the base for energy calculation. For a 1-Ω resistor, $p(t) = v^2(t) = i^2(t) = f^2(t)$, where $f(t)$ stands for either voltage or current. The energy delivered to the 1-Ω resistor is

$$W_{1\Omega} = \int_{-\infty}^{\infty} f^2(t)\, dt \tag{17.58}$$

Parseval's theorem states that this same energy can be calculated in the frequency domain as

$$W_{1\Omega} = \int_{-\infty}^{\infty} f^2(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\, d\omega \tag{17.59}$$

> **Parseval's theorem** states that the total energy delivered to a 1-Ω resistor equals the total area under the square of $f(t)$ or $1/2\pi$ times the total area under the square of the magnitude of the Fourier transform of $f(t)$.

Parseval's theorem relates energy associated with a signal to its Fourier transform. It provides the physical significance of $F(\omega)$, namely, that $|F(\omega)|^2$ is a measure of the energy density (in joules per hertz) corresponding to $f(t)$.

In fact, $|F(\omega)|^2$ is sometimes known as the energy spectral density of signal $f(t)$.

To derive Eq. (17.59), we begin with Eq. (17.58) and substitute Eq. (17.9) for one of the $f(t)$'s. We obtain

$$W_{1\Omega} = \int_{-\infty}^{\infty} f^2(t)\, dt = \int_{-\infty}^{\infty} f(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t} d\omega\right] dt \tag{17.60}$$

The function $f(t)$ can be moved inside the integral within the brackets, since the integral does not involve time:

$$W_{1\Omega} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)F(\omega)e^{j\omega t}\, d\omega\, dt \tag{17.61}$$

Reversing the order of integration,

$$\begin{aligned} W_{1\Omega} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\left[\int_{-\infty}^{\infty} f(t)e^{-j(-\omega)t}\, dt\right] d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)F(-\omega)\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)F^*(\omega)\, d\omega \end{aligned} \tag{17.62}$$

But if $z = x + jy$, $zz^* = (x + jy)(x - jy) = x^2 + y^2 = |z|^2$. Hence,

$$W_{1\Omega} = \int_{-\infty}^{\infty} f^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\,d\omega \tag{17.63}$$

as expected. Equation (17.63) indicates that the energy carried by a signal can be found by integrating either the square of $f(t)$ in the time domain or $1/2\pi$ times the square of $F(\omega)$ in the frequency domain.

Since $|F(\omega)|^2$ is an even function, we may integrate from 0 to ∞ and double the result, that is,

$$W_{1\Omega} = \int_{-\infty}^{\infty} f^2(t)\,dt = \frac{1}{\pi}\int_{0}^{\infty} |F(\omega)|^2\,d\omega \tag{17.64}$$

We may also calculate the energy in any frequency band $\omega_1 < \omega < \omega_2$ as

$$W_{1\Omega} = \frac{1}{\pi}\int_{\omega_1}^{\omega_2} |F(\omega)|^2\,d\omega \tag{17.65}$$

Notice that Parseval's theorem as stated here applies to nonperiodic functions. Parseval's theorem for periodic functions was presented in Sections 16.5 and 16.6. As evident in Eq. (17.63), Parseval's theorem shows that the energy associated with a nonperiodic signal is spread over the entire frequency spectrum, whereas the energy of the periodic signal is concentrated at the frequencies of its harmonic components.

EXAMPLE 17.9

The voltage across a 10-Ω resistor is $v(t) = 5e^{-3t}u(t)$ V. Find the total energy dissipated in the resistor.

Solution:

We can find the energy using either $f(t) = v(t)$ or $F(\omega) = V(\omega)$. In the time domain,

$$W_{10\Omega} = 10\int_{-\infty}^{\infty} f^2(t)\,dt = 10\int_{0}^{\infty} 25e^{-6t}\,dt$$

$$= 250\frac{e^{-6t}}{-6}\bigg|_0^{\infty} = \frac{250}{6} = 41.67 \text{ J}$$

In the frequency domain,

$$F(\omega) = V(\omega) = \frac{5}{3 + j\omega}$$

so that

$$|F(\omega)|^2 = F(\omega)F^*(\omega) = \frac{25}{9 + \omega^2}$$

Hence, the energy dissipated is

$$W_{10\Omega} = \frac{10}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\,d\omega = \frac{10}{\pi}\int_{0}^{\infty} \frac{25}{9 + \omega^2}\,d\omega$$

$$= \frac{250}{\pi}\left(\frac{1}{3}\tan^{-1}\frac{\omega}{3}\right)\bigg|_0^{\infty} = \frac{250}{\pi}\left(\frac{1}{3}\right)\left(\frac{\pi}{2}\right) = \frac{250}{6} = 41.67 \text{ J}$$

PRACTICE PROBLEM 17.9

(a) Calculate the total energy absorbed by a 1-Ω resistor with $i(t) = 10e^{-2|t|}$ A in the time domain. (b) Repeat (a) in the frequency domain.

Answer: (a) 50 J, (b) 50 J.

EXAMPLE 17.10

Calculate the fraction of the total energy dissipated by a 1-Ω resistor in the frequency band $0 < \omega < 10$ rad/s when the voltage across it is $v(t) = e^{-2t}u(t)$.

Solution:

Given that $f(t) = v(t) = e^{-2t}u(t)$, then

$$F(\omega) = \frac{1}{2 + j\omega} \qquad \Longrightarrow \qquad |F(\omega)|^2 = \frac{1}{4 + \omega^2}$$

The total energy dissipated by the resistor is

$$W_{1\Omega} = \frac{1}{\pi}\int_0^\infty |F(\omega)|^2\,d\omega = \frac{1}{\pi}\int_0^\infty \frac{d\omega}{4+\omega^2}$$

$$= \frac{1}{\pi}\left(\frac{1}{2}\tan^{-1}\frac{\omega}{2}\Big|_0^\infty\right) = \frac{1}{\pi}\left(\frac{1}{2}\right)\frac{\pi}{2} = 0.25\text{ J}$$

The energy in the frequencies $0 < \omega < 10$ is

$$W = \frac{1}{\pi}\int_0^{10} |F(\omega)|^2 d\omega = \frac{1}{\pi}\int_0^{10}\frac{d\omega}{4+\omega^2} = \frac{1}{\pi}\left(\frac{1}{2}\tan^1\frac{\omega}{2}\Big|_0^{10}\right)$$

$$= \frac{1}{2\pi}\tan^{-1}5 = \frac{1}{2\pi}\left(\frac{78.69^\circ}{180^\circ}\pi\right) = 0.218\text{ J}$$

Its percentage of the total energy is

$$\frac{W}{W_{1\Omega}} = \frac{0.218}{0.25} = 87.4\ \%$$

PRACTICE PROBLEM 17.10

A 2-Ω resistor has $i(t) = e^{-t}u(t)$. What percentage of the total energy is in the frequency band $-4 < \omega < 4$ rad/s?

Answer: 84.4 percent.

17.6 COMPARING THE FOURIER AND LAPLACE TRANSFORMS

It is worthwhile to take some moments to compare the Laplace and Fourier transforms. The following similarities and differences should be noted:

1. The Laplace transform defined in Chapter 14 is one-sided in that the integral is over $0 < t < \infty$, making it only useful for positive-time functions, $f(t)$, $t > 0$. The Fourier transform is applicable to functions defined for all time.

2. For a function $f(t)$ that is nonzero for positive time only (i.e., $f(t) = 0,\ \ t < 0$) and $\int_0^\infty |f(t)|\,dt < \infty$, the two transforms are related by

$$F(\omega) = F(s)\Big|_{s=j\omega} \tag{17.66}$$

This equation also shows that the Fourier transform can be regarded as a special case of the Laplace transform with $s = j\omega$. Recall that $s = \sigma + j\omega$. Therefore, Eq. (17.66) shows that the Laplace transform is related to the entire s plane, whereas the Fourier transform is restricted to the $j\omega$ axis. See Fig. 15.1.

3. The Laplace transform is applicable to a wider range of functions than the Fourier transform. For example, the function $tu(t)$ has a Laplace transform but no Fourier transform. But Fourier transforms exist for signals that are not physically realizable and have no Laplace transforms.
4. The Laplace transform is better suited for the analysis of transient problems involving initial conditions, since it permits the inclusion of the initial conditions, whereas the Fourier transform does not. The Fourier transform is especially useful for problems in the steady state.
5. The Fourier transform provides greater insight into the frequency characteristics of signals than does the Laplace transform.

In other words, if all the poles of $F(s)$ lie in the left-hand side of the s plane, then one can obtain the Fourier transform $F(\omega)$ from the corresponding Laplace transform $F(s)$ by merely replacing s by $j\omega$. Note that this is not the case, for example, for $u(t)$ or $\cos at u(t)$.

Some of the similarities and differences can be observed by comparing Tables 15.1 and 15.2 with Tables 17.1 and 17.2.

†17.7 APPLICATIONS

Besides its usefulness for circuit analysis, the Fourier transform is used extensively in a variety of fields such as optics, spectroscopy, acoustics, computer science, and electrical engineering. In electrical engineering, it is applied in communications systems and signal processing, where frequency response and frequency spectra are vital. Here we consider two simple applications: amplitude modulation (AM) and sampling.

17.7.1 Amplitude Modulation

Electromagnetic radiation or transmission of information through space has become an indispensable part of a modern technological society. However, transmission through space is only efficient and economical at radio frequencies (above 20 kHz). To transmit intelligent signals—such as for speech and music—contained in the low-frequency range of 50 Hz to 20 kHz is expensive; it requires a huge amount of power and large antennas. A common method of transmitting low-frequency audio information is to transmit a high-frequency signal, called a *carrier*, which is controlled in some way to correspond to the audio information. Three characteristics (amplitude, frequency, or phase) of a carrier can be controlled so as to allow it to carry the intelligent signal, called the *modulating signal*. Here we will only consider the control of the carrier's amplitude. This is known as *amplitude modulation.*

Amplitude modulation (AM) is a process whereby the amplitude of the carrier is controlled by the modulating signal.

AM is used in ordinary commercial radio bands and the video portion of commercial television.

Suppose the audio information, such as voice or music (or the modulating signal in general) to be transmitted is $m(t) = V_m \cos \omega_m t$, while the high-frequency carrier is $c(t) = V_c \cos \omega_c t$, where $\omega_c >> \omega_m$. Then an AM signal $f(t)$ is given by

$$f(t) = V_c[1 + m(t)] \cos \omega_c t \tag{17.67}$$

Figure 17.22 illustrates the modulating signal $m(t)$, the carrier $c(t)$, and the AM signal $f(t)$. We can use the result in Eq. (17.27) together with the Fourier transform of the cosine function (see Example 17.1 or Table 17.1) to determine the spectrum of the AM signal:

$$\begin{aligned} F(\omega) &= \mathcal{F}[V_c \cos \omega_c t] + \mathcal{F}[V_c m(t) \cos \omega_c t] \\ &= V_c \pi[\delta(\omega - \omega_c) + \delta(\omega + \omega_c)] \\ &\quad + \frac{V_c}{2}[M(\omega - \omega_c) + M(\omega + \omega_c)] \end{aligned} \tag{17.68}$$

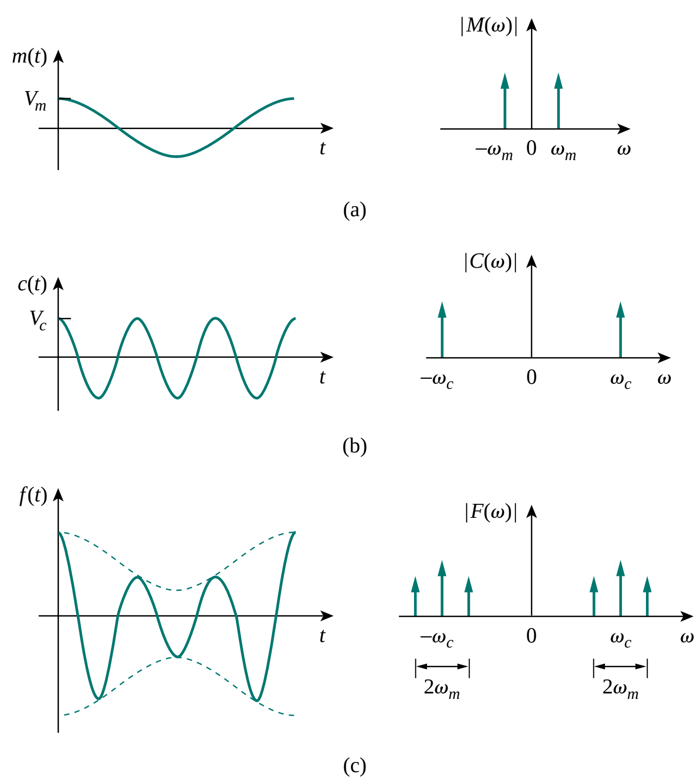

Figure 17.22 Time domain and frequency display of: (a) modulating signal, (b) carrier signal, (c) AM signal.

where $M(\omega)$ is the Fourier transform of the modulating signal $m(t)$. Shown in Fig. 17.23 is the frequency spectrum of the AM signal. Figure 17.23 indicates that the AM signal consists of the carrier and two other sinusoids. The sinusoid with frequency $\omega_c - \omega_m$ is known as the *lower sideband*, while the one with frequency $\omega_c + \omega_m$ is known as the *upper sideband*.

Notice that we have assumed that the modulating signal is sinusoidal to make the analysis easy. In real life, $m(t)$ is a nonsinusoidal, band-limited signal—its frequency spectrum is within the range between 0 and $\omega_u = 2\pi f_u$ (i.e., the signal has an upper frequency limit). Typically, $f_u = 5$ kHz for AM radio. If the frequency spectrum of the modulating signal is as shown in Fig. 17.24(a), then the frequency spectrum of the AM signal is shown in Fig. 17.24(b). Thus, to avoid any interference, carriers for AM radio stations are spaced 10 kHz apart.

At the receiving end of the transmission, the audio information is recovered from the modulated carrier by a process known as *demodulation*.

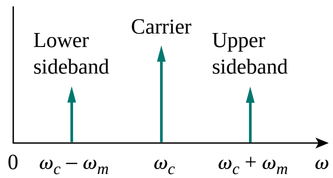

Figure 17.23 Frequency spectrum of AM signal.

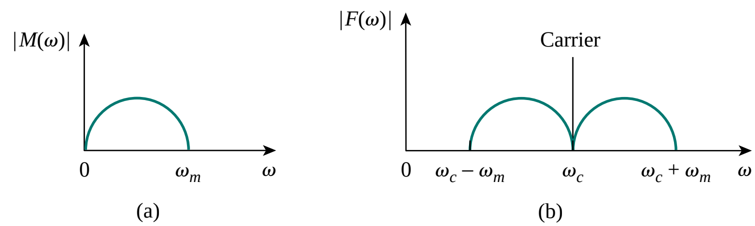

Figure 17.24 Frequency spectrum of: (a) modulating signal, (b) AM signal.

EXAMPLE 17.11

A music signal has frequency components from 15 Hz to 30 kHz. If this signal could be used to amplitude modulate a 1.2-MHz carrier, find the range of frequencies for the lower and upper sidebands.

Solution:

The lower sideband is the difference of the carrier and modulating frequencies. It will include the frequencies from

$$1{,}200{,}000 - 30{,}000 \text{ Hz} = 1{,}170{,}000 \text{ Hz}$$

to

$$1{,}200{,}000 - 15 \text{ Hz} = 1{,}199{,}985 \text{ Hz}$$

The upper sideband is the sum of the carrier and modulating frequencies. It will include the frequencies from

$$1{,}200{,}000 + 15 \text{ Hz} = 1{,}200{,}015 \text{ Hz}$$

to

$$1{,}200{,}000 + 30{,}000 \text{ Hz} = 1{,}230{,}000 \text{ Hz}$$

PRACTICE PROBLEM 17.11

If a 2-MHz carrier is modulated by a 4-kHz intelligent signal, determine the frequencies of the three components of the AM signal that results.

Answer: 2,004,000 Hz, 2,000,000 Hz, 1,996,000 Hz.

17.7.2 Sampling

In analog systems, signals are processed in their entirety. However, in modern digital systems, only samples of signals are required for processing. This is possible as a result of the sampling theorem given in Section 16.8.1. The sampling can be done by using a train of pulses or impulses. We will use impulse sampling here.

Consider the continuous signal $g(t)$ shown in Fig. 17.25(a). This can be multiplied by a train of impulses $\delta(t - nT_s)$ shown in Fig. 17.25(b), where T_s is the *sampling interval* and $f_s = 1/T_s$ is the *sampling frequency* or the *sampling rate*. The sampled signal $g_s(t)$ is therefore

$$g_s(t) = g(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} g(nT_s)\delta(t - nT_s) \qquad (17.69)$$

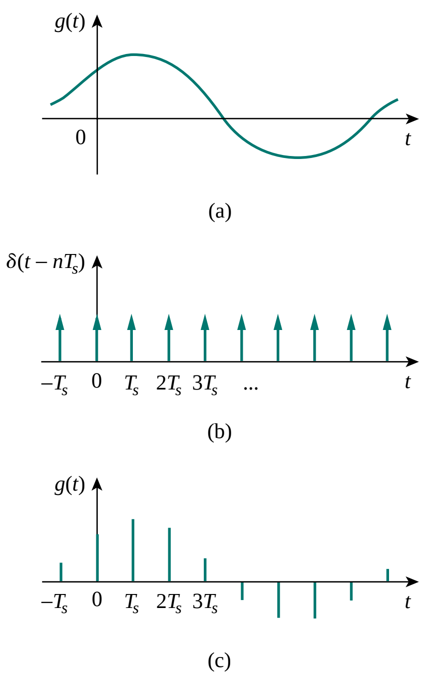

Figure 17.25 (a) Continuous (analog) signal to be sampled, (b) train of impulses, (c) sampled (digital) signal.

The Fourier transform of this is

$$G_s(\omega) = \sum_{n=-\infty}^{\infty} g(nT_s)\mathcal{F}[\delta(t - nT_s)] = \sum_{n=-\infty}^{\infty} g(nT_s)e^{-jn\omega T_s} \qquad (17.70)$$

It can be shown that

$$\sum_{n=-\infty}^{\infty} g(nT_s)e^{-jn\omega T_s} = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} G(\omega + n\omega_s) \qquad (17.71)$$

where $\omega_s = 2\pi/T_s$. Thus, Eq. (17.70) becomes

$$G_s(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} G(\omega + n\omega_s) \qquad (17.72)$$

This shows that the Fourier transform $G_s(\omega)$ of the sampled signal is a sum of translates of the Fourier transform of the original signal at a rate of $1/T_s$.

In order to ensure optimum recovery of the original signal, what must be the sampling interval? This fundamental question in sampling is answered by an equivalent part of the sampling theorem:

> A band-limited signal, with no frequency component higher than W hertz, may be completely recovered from its samples taken at a frequency at least twice as high as 2W samples per second.

In other words, for a signal with bandwidth W hertz, there is no loss of information or overlapping if the sampling frequency is at least twice the highest frequency in the modulating signal. Thus,

$$\frac{1}{T_s} = f_s \geq 2W \qquad (17.73)$$

The sampling frequency $f_s = 2W$ is known as the *Nyquist* frequency or rate, and $1/f_s$ is the Nyquist interval.

EXAMPLE 17.12

A telephone signal with a cutoff frequency of 5 kHz is sampled at a rate 60 percent higher than the minimum allowed rate. Find the sampling rate.

Solution:

The minimum sample rate is the Nyquist rate $= 2W = 2 \times 5 =$ 10 kHz. Hence,

$$f_s = 1.60 \times 2W = 16 \text{ kHz}$$

PRACTICE PROBLEM 17.12

An audio signal that is band-limited to 12.5 kHz is digitized into 8-bit samples. What is the maximum sampling interval that must be used to ensure complete recovery?

Answer: 40 μs.

17.8 SUMMARY

1. The Fourier transform converts a nonperiodic function $f(t)$ into a transform $F(\omega)$ where

$$F(\omega) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$

2. The inverse Fourier transform of $F(\omega)$ is

$$f(t) = \mathcal{F}^{-1}[F(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega$$

3. Important Fourier transform properties and pairs are summarized in Tables 17.1 and 17.2, respectively.
4. Using the Fourier transform method to analyze a circuit involves finding the Fourier transform of the excitation, transforming the circuit element into the frequency domain, solving for the unknown response, and transforming the response to the time domain using the inverse Fourier transform.
5. If $H(\omega)$ is the transfer function of a network, then $H(\omega)$ is the Fourier transform of the network's impulse response; that is,

$$H(\omega) = \mathcal{F}[h(t)]$$

The output $V_o(\omega)$ of the network can be obtained from the input $V_i(\omega)$ using

$$V_o(\omega) = H(\omega)V_i(\omega)$$

6. Parseval's theorem gives the energy relationship between a function $f(t)$ and its Fourier transform $F(\omega)$. The 1-Ω energy is

$$W_{1\Omega} = \int_{-\infty}^{\infty} f^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\,d\omega$$

The theorem is useful in calculating energy carried by a signal either in the time domain or in the frequency domain.

7. Typical applications of the Fourier transform are found in amplitude modulation (AM) and sampling. For AM application, a way of determining the sidebands in an amplitude-modulated wave is derived from the modulation property of the Fourier transform. For sampling application, we found that no information is lost in sampling (required for digital transmission) if the sampling frequency is at least twice equal to the Nyquist rate.

REVIEW QUESTIONS

17.1 Which of these functions does not have a Fourier transform?
(a) $e^t u(-t)$ (b) $te^{-3t}u(t)$
(c) $1/t$ (d) $|t|u(t)$

17.2 The Fourier transform of e^{j2t} is:
(a) $\dfrac{1}{2+j\omega}$ (b) $\dfrac{1}{-2+j\omega}$
(c) $2\pi\delta(\omega-2)$ (d) $2\pi\delta(\omega+2)$

17.3 The inverse Fourier transform of $\dfrac{e^{-j\omega}}{2+j\omega}$ is
(a) e^{-2t} (b) $e^{-2t}u(t-1)$
(c) $e^{-2(t-1)}$ (d) $e^{-2(t-1)}u(t-1)$

17.4 The inverse Fourier transform of $\delta(\omega)$ is:
(a) $\delta(t)$ (b) $u(t)$ (c) 1 (d) $1/2\pi$

17.5 The inverse Fourier transform of $j\omega$ is:
(a) $1/t$ (b) $\delta'(t)$
(c) $u'(t)$ (d) undefined

17.6 Evaluating the integral $\displaystyle\int_{-\infty}^{\infty} \frac{10\delta(\omega)}{4+\omega^2}\,d\omega$ results in:
(a) 0 (b) 2 (c) 2.5 (d) ∞

17.7 The integral $\displaystyle\int_{-\infty}^{\infty} \frac{10\delta(\omega-1)}{4+\omega^2}\,d\omega$ gives:
(a) 0 (b) 2 (c) 2.5 (d) ∞

17.8 The current through a 1-F capacitor is $\delta(t)$ A. The voltage across the capacitor is:
(a) $u(t)$ (b) $-1/2+u(t)$
(c) $e^{-t}u(t)$ (d) $\delta(t)$

17.9 A unit step current is applied through a 1-H inductor. The voltage across the inductor is:
(a) $u(t)$ (b) $\text{sgn}(t)$
(c) $e^{-t}u(t)$ (d) $\delta(t)$

17.10 Parseval's theorem is only for nonperiodic functions.
(a) True (b) False

Answers: 17.1c, 17.2c, 17.3d, 17.4d, 17.5b, 17.6c, 17.7b, 17.8b, 17.9d, 17.10b

PROBLEMS

Sections 17.2 and 17.3 Fourier Transform and its Properties

17.1 Obtain the Fourier transform of the function in Fig. 17.26.

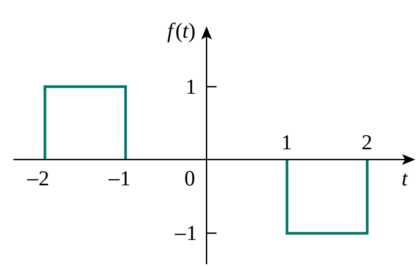

Figure 17.26 For Prob. 17.1.

17.2 What is the Fourier transform of the triangular pulse in Fig. 17.27?

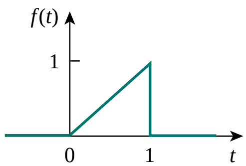

Figure 17.27 For Prob. 17.2.

17.3 Calculate the Fourier transform of the signal in Fig. 17.28.

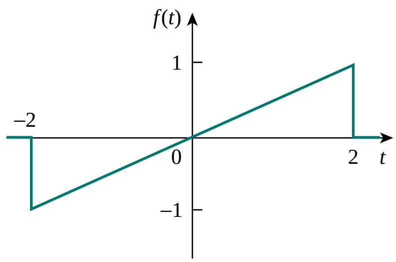

Figure 17.28 For Prob. 17.3.

17.4 Find the Fourier transforms of the signals in Fig. 17.29.

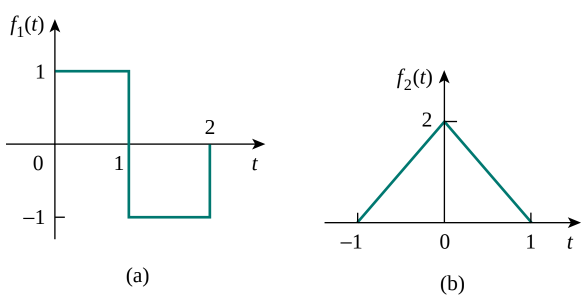

Figure 17.29 For Prob. 17.4.

17.5 Determine the Fourier transforms of the functions in Fig. 17.30.

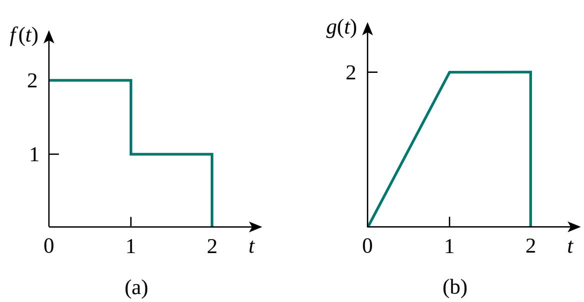

Figure 17.30 For Prob. 17.5.

17.6 Obtain the Fourier transforms of the signals shown in Fig. 17.31.

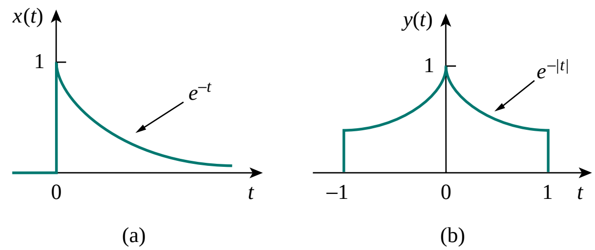

Figure 17.31 For Prob. 17.6.

17.7 Find the Fourier transform of the "sine-wave pulse" shown in Fig. 17.32.

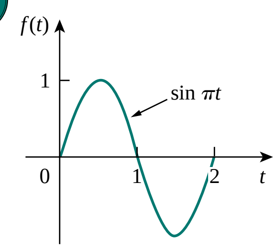

Figure 17.32 For Prob. 17.7.

17.8 Determine the Fourier transforms of these functions:

(a) $f(t) = e^t[u(t) - u(t-1)]$

(b) $g(t) = te^{-t}u(t)$

(c) $h(t) = u(t+1) - 2u(t) + u(t-1)$

17.9 Find the Fourier transforms of these functions:

(a) $f(t) = e^{-t}\cos(3t+\pi)u(t)$

(b) $g(t) = \sin \pi t[u(t+1) - u(t-1)]$

(c) $h(t) = e^{-2t}\cos \pi t u(t-1)$

(d) $p(t) = e^{-2t}\sin 4t u(-t)$

(e) $q(t) = 4\,\mathrm{sgn}(t-2) + 3\delta(t) - 2u(t-2)$

17.10 Find the Fourier transforms of the following functions:

(a) $f(t) = \delta(t+3) - \delta(t-3)$

(b) $f(t) = \int_{-\infty}^{\infty} 2\delta(t-1)\,dt$

(c) $f(t) = \delta(3t) - \delta'(2t)$

***17.11** Determine the Fourier transforms of these functions:

(a) $f(t) = 4/t^2$ (b) $g(t) = 8/(4+t^2)$

17.12 Find the Fourier transforms of:

(a) $\cos 2t u(t)$ (b) $\sin 10t u(t)$

17.13 Obtain the Fourier transform of $y(t) = e^{-t}\cos t u(t)$.

*An asterisk indicates a challenging problem.

17.14 Find the Fourier transform of $f(t) = \cos 2\pi t[u(t) - u(t-1)]$.

17.15 (a) Show that a periodic signal with exponential Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

has the Fourier transform

$$F(\omega) = \sum_{n=-\infty}^{\infty} c_n \delta(\omega - n\omega_0)$$

where $\omega_0 = 2\pi/T$.

(b) Find the Fourier transform of the signal in Fig. 17.33.

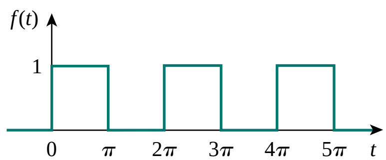

Figure 17.33 For Prob. 17.15(b).

17.16 Prove that if $F(\omega)$ is the Fourier transform of $f(t)$,

$$\mathcal{F}[f(t)\sin\omega_0 t] = \frac{j}{2}[F(\omega+\omega_0) - F(\omega-\omega_0)]$$

17.17 If the Fourier transform of $f(t)$ is

$$F(\omega) = \frac{10}{(2+j\omega)(5+j\omega)}$$

determine the transforms of the following:

(a) $f(-3t)$ (b) $f(2t-1)$ (c) $f(t)\cos 2t$

(d) $\frac{d}{dt}f(t)$ (e) $\int_{-\infty}^{t} f(t)\,dt$

17.18 Given that $\mathcal{F}[f(t)] = (j/\omega)(e^{-j\omega} - 1)$, find the Fourier transforms of:

(a) $x(t) = f(t) + 3$ (b) $y(t) = f(t-2)$

(c) $h(t) = f'(t)$

(d) $g(t) = 4f(\frac{2}{3}t) + 10f(\frac{5}{3}t)$

17.19 Obtain the inverse Fourier transforms of:

(a) $F(\omega) = \dfrac{10}{j\omega(j\omega+2)}$

(b) $F(\omega) = \dfrac{4-j\omega}{\omega^2 - 3j\omega - 2}$

17.20 Find the inverse Fourier transforms of the following functions:

(a) $F(\omega) = \dfrac{100}{j\omega(j\omega+10)}$

(b) $G(\omega) = \dfrac{10j\omega}{(-j\omega+2)(\omega+3)}$

(c) $H(\omega) = \dfrac{60}{-\omega^2 + j40\omega + 1300}$

(d) $Y(\omega) = \dfrac{\delta(\omega)}{(j\omega+1)(j\omega+2)}$

17.21 Find the inverse Fourier transforms of:

(a) $\dfrac{\pi\delta(\omega)}{(5+j\omega)(2+j\omega)}$

(b) $\dfrac{10\delta(\omega+2)}{j\omega(j\omega+1)}$

(c) $\dfrac{20\delta(\omega-1)}{(2+j\omega)(3+j\omega)}$

(d) $\dfrac{5\pi\delta(\omega)}{5+j\omega} + \dfrac{5}{j\omega(5+j\omega)}$

***17.22** Determine the inverse Fourier transforms of:

(a) $F(\omega) = 4\delta(\omega+3) + \delta(\omega) + 4\delta(\omega-3)$

(b) $G(\omega) = 4u(\omega+2) - 4u(\omega-2)$

(c) $H(\omega) = 6\cos 2\omega$

***17.23** Determine the functions corresponding to the following Fourier transforms:

(a) $F_1(\omega) = \dfrac{e^{j\omega}}{-j\omega+1}$ (b) $F_2(\omega) = 2e^{|\omega|}$

(c) $F_3(\omega) = \dfrac{1}{(1+\omega^2)^2}$ (d) $F_4(\omega) = \dfrac{\delta(\omega)}{1+j2\omega}$

***17.24** Find $f(t)$ if:

(a) $F(\omega) = 2\sin\pi\omega[u(\omega+1) - u(\omega-1)]$

(b) $F(\omega) = \dfrac{1}{\omega}(\sin 2\omega - \sin\omega) + \dfrac{j}{\omega}(\cos 2\omega - \cos\omega)$

17.25 Determine the signal $f(t)$ whose Fourier transform is shown in Fig. 17.34. (*Hint:* Use the duality property.)

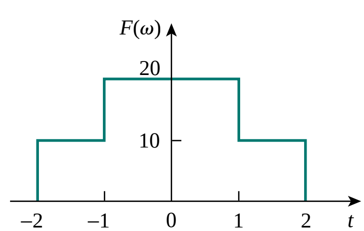

Figure 17.34 For Prob. 17.25.

Section 17.4 Circuit Applications

17.26 A linear system has a transfer function

$$H(\omega) = \frac{10}{2+j\omega}$$

Determine the output $v_o(t)$ at $t = 2$ s if the input $v_i(t)$ equals:

(a) $4\delta(t)$ V (b) $6e^{-t}u(t)$ V (c) $3\cos 2t$ V

17.27 Find the transfer function $I_o(\omega)/I_s(\omega)$ for the circuit in Fig. 17.35.

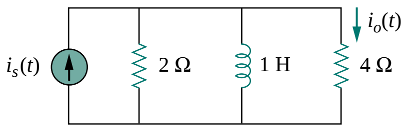

Figure 17.35 For Prob. 17.27.

17.28 Obtain $v_o(t)$ in the circuit of Fig. 17.36 when $v_i(t) = u(t)$ V.

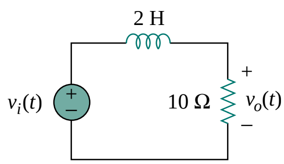

Figure 17.36 For Prob. 17.28.

17.29 Determine the current $i(t)$ in the circuit of Fig. 17.37(b), given the voltage source shown in Fig. 17.37(a).

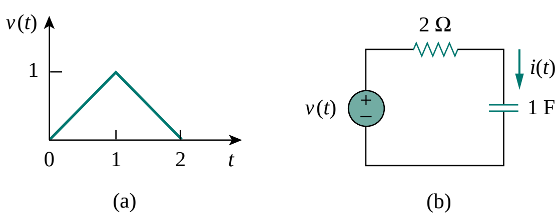

Figure 17.37 For Prob. 17.29.

17.30 Obtain the current $i_o(t)$ in the circuit in Fig. 17.38.
(a) Let $i(t) = \text{sgn}(t)$ A.
(b) Let $i(t) = 4[u(t) - u(t-1)]$ A.

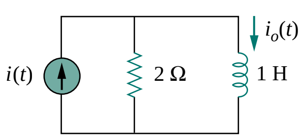

Figure 17.38 For Prob. 17.30.

17.31 Find current $i_o(t)$ in the circuit of Fig. 17.39.

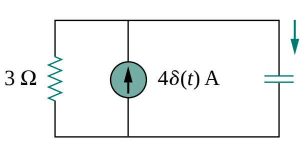

Figure 17.39 For Prob. 17.31.

17.32 If the rectangular pulse in Fig. 17.40(a) is applied to the circuit in Fig. 17.40(b), find v_o at $t = 1$ s.

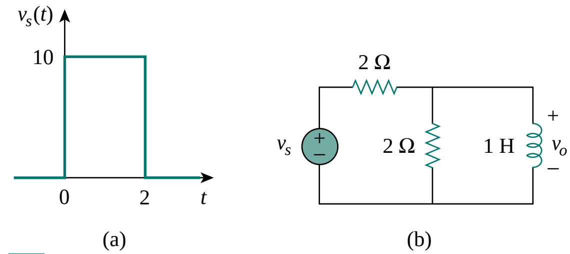

Figure 17.40 For Prob. 17.32.

***17.33** Calculate $v_o(t)$ in the circuit of Fig. 17.41 if $v_s(t) = 10e^{-|t|}$ V.

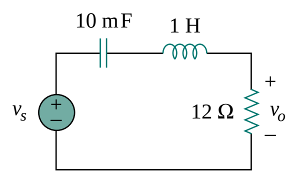

Figure 17.41 For Prob. 17.33.

17.34 Determine the Fourier transform of $i_o(t)$ in the circuit of Fig. 17.42.

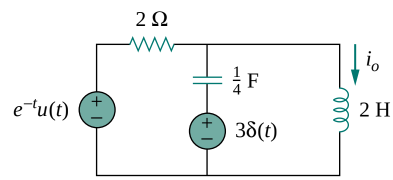

Figure 17.42 For Prob. 17.34.

17.35 In the circuit of Fig. 17.43, let $i_s = 4\delta(t)$ A. Find $V_o(\omega)$.

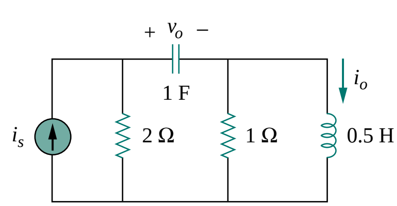

Figure 17.43 For Prob. 17.35.

17.36 Find $i_o(t)$ in the op amp circuit of Fig. 17.44.

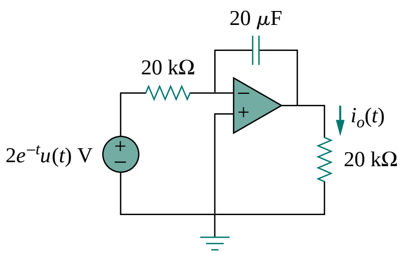

Figure 17.44 For Prob. 17.36.

17.37 Use the Fourier transform method to obtain $v_o(t)$ in the circuit of Fig. 17.45.

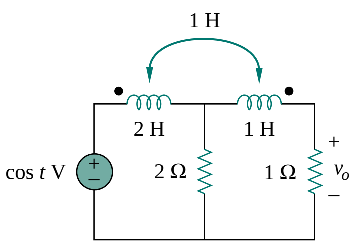

Figure 17.45 For Prob. 17.37.

17.38 Determine $v_o(t)$ in the transformer circuit of Fig. 17.46.

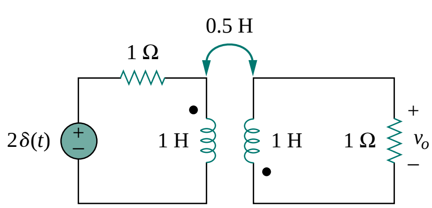

Figure 17.46 For Prob. 17.38.

Section 17.5 Parseval's Theorem

17.39 For $F(\omega) = \dfrac{1}{3 + j\omega}$, find $J = \int_{-\infty}^{\infty} f^2(t)\,dt$.

17.40 If $f(t) = e^{-2|t|}$, find $J = \int_{-\infty}^{\infty} |F(\omega)|^2\,d\omega$.

17.41 Given the signal $f(t) = 4e^{-t}u(t)$, what is the total energy in $f(t)$?

17.42 Let $f(t) = 5e^{-(t-2)}u(t)$. Find $F(\omega)$ and use it to find the total energy in $f(t)$.

17.43 A voltage source $v_s(t) = e^{-t}\sin 2t\; u(t)$ V is applied to a 1-Ω resistor. Calculate the energy delivered to the resistor.

17.44 Let $i(t) = 2e^{t}u(-t)$ A. Find the total energy carried by $i(t)$ and the percentage of the 1-Ω energy in the frequency range of $-5 < \omega < 5$ rad/s.

Section 17.6 Applications

17.45 An AM signal is specified by

$$f(t) = 10(1 + 4\cos 200\pi t)\cos \pi \times 10^4 t$$

Determine the following:

(a) the carrier frequency,

(b) the lower sideband frequency,

(c) the upper sideband frequency.

17.46 A carrier wave of frequency 8 MHz is amplitude-modulated by a 5-kHz signal. Determine the lower and upper sidebands.

17.47 A voice signal occupying the frequency band of 0.4 to 3.5 kHz is used to amplitude-modulate a 10-MHz carrier. Determine the range of frequencies for the lower and upper sidebands.

17.48 For a given locality, calculate the number of stations allowable in the AM broadcasting band (540 to 1600 kHz) without interference with one another.

17.49 Repeat the previous problem for the FM broadcasting band (88 to 108 MHz), assuming that the carrier frequencies are spaced 200 kHz apart.

17.50 The highest-frequency component of a voice signal is 3.4 kHz. What is the Nyquist rate of the sampler of the voice signal?

17.51 A TV signal is band-limited to 4.5 MHz. If samples are to be reconstructed at a distant point, what is the maximum sampling interval allowable?

***17.52** Given a signal $g(t) =$ sinc $(200\pi t)$, find the Nyquist rate and the Nyquist interval for the signal.

COMPREHENSIVE PROBLEMS

17.53 The voltage signal at the input of a filter is $v(t) = 50e^{-2|t|}$ V. What percentage of the total 1-Ω energy content lies in the frequency range of $1 < \omega < 5$ rad/s?

17.54 A signal with Fourier transform

$$F(\omega) = \frac{20}{4 + j\omega}$$

is passed through a filter whose cutoff frequency is 2 rad/s (i.e., $0 < \omega < 2$). What fraction of the energy in the input signal is contained in the output signal?

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