CHAPTER 6

CAPACITORS AND INDUCTORS

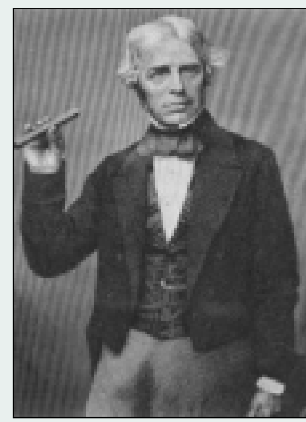
The important thing about a problem is not its solution, but the strength we gain in finding the solution.

—Anonymous

Historical Profiles

Michael Faraday (1791–1867), an English chemist and physicist, was probably the greatest experimentalist who ever lived.

Born near London, Faraday realized his boyhood dream by working with the great chemist Sir Humphry Davy at the Royal Institution, where he worked for 54 years. He made several contributions in all areas of physical science and coined such words as electrolysis, anode, and cathode. His discovery of electromagnetic induction in 1831 was a major breakthrough in engineering because it provided a way of generating electricity. The electric motor and generator operate on this principle. The unit of capacitance, the farad, was named in his honor.



Joseph Henry (1797–1878), an American physicist, discovered inductance and constructed an electric motor.

Born in Albany, New York, Henry graduated from Albany Academy and taught philosophy at Princeton University from 1832 to 1846. He was the first secretary of the Smithsonian Institution. He conducted several experiments on electromagnetism and developed powerful electromagnets that could lift objects weighing thousands of pounds. Interestingly, Joseph Henry discovered electromagnetic induction before Faraday but failed to publish his findings. The unit of inductance, the henry, was named after him.

6.1 INTRODUCTION

In contrast to a resistor, which spends or dissipates energy irreversibly, an inductor or capacitor stores or releases energy (i.e., has a memory).

So far we have limited our study to resistive circuits. In this chapter, we shall introduce two new and important passive linear circuit elements: the capacitor and the inductor. Unlike resistors, which dissipate energy, capacitors and inductors do not dissipate but store energy, which can be retrieved at a later time. For this reason, capacitors and inductors are called *storage* elements.

The application of resistive circuits is quite limited. With the introduction of capacitors and inductors in this chapter, we will be able to analyze more important and practical circuits. Be assured that the circuit analysis techniques covered in Chapters 3 and 4 are equally applicable to circuits with capacitors and inductors.

We begin by introducing capacitors and describing how to combine them in series or in parallel. Later, we do the same for inductors. As typical applications, we explore how capacitors are combined with op amps to form integrators, differentiators, and analog computers.

6.2 CAPACITORS

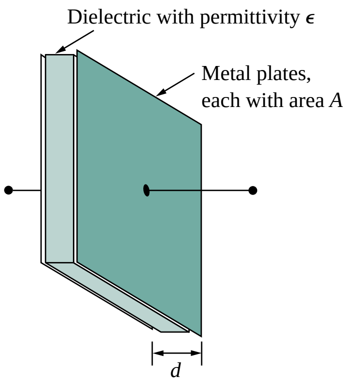


Figure 6.1 A typical capacitor.

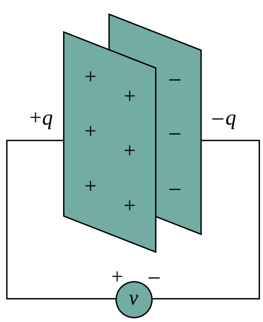


Figure 6.2 A capacitor with applied voltage v.

A capacitor is a passive element designed to store energy in its electric field. Besides resistors, capacitors are the most common electrical components. Capacitors are used extensively in electronics, communications, computers, and power systems. For example, they are used in the tuning circuits of radio receivers and as dynamic memory elements in computer systems.

A capacitor is typically constructed as depicted in Fig. 6.1.

> A **capacitor** consists of two conducting plates separated by an insulator (or dielectric).

In many practical applications, the plates may be aluminum foil while the dielectric may be air, ceramic, paper, or mica.

When a voltage source v is connected to the capacitor, as in Fig. 6.2, the source deposits a positive charge q on one plate and a negative charge $-q$ on the other. The capacitor is said to store the electric charge. The amount of charge stored, represented by q, is directly proportional to the applied voltage v so that

$$\boxed{q = Cv} \tag{6.1}$$

where C, the constant of proportionality, is known as the *capacitance* of the capacitor. The unit of capacitance is the farad (F), in honor of the English physicist Michael Faraday (1791–1867). From Eq. (6.1), we may derive the following definition.

Alternatively, capacitance is the amount of charge stored per plate for a unit voltage difference in a capacitor.

> **Capacitance** is the ratio of the charge on one plate of a capacitor to the voltage difference between the two plates, measured in farads (F).

Note from Eq. (6.1) that 1 farad = 1 coulomb/volt.

Although the capacitance C of a capacitor is the ratio of the charge q per plate to the applied voltage v, it does not depend on q or v. It depends on the physical dimensions of the capacitor. For example, for the parallel-plate capacitor shown in Fig. 6.1, the capacitance is given by

$$C = \frac{\epsilon A}{d} \tag{6.2}$$

Capacitor voltage rating and capacitance are typically inversely rated due to the relationships in Eqs. (6.1) and (6.2). Arcing occurs if d is small and V is high.

where A is the surface area of each plate, d is the distance between the plates, and ϵ is the permittivity of the dielectric material between the plates. Although Eq. (6.2) applies to only parallel-plate capacitors, we may infer from it that, in general, three factors determine the value of the capacitance:

1. The surface area of the plates—the larger the area, the greater the capacitance.
2. The spacing between the plates—the smaller the spacing, the greater the capacitance.
3. The permittivity of the material—the higher the permittivity, the greater the capacitance.

Capacitors are commercially available in different values and types. Typically, capacitors have values in the picofarad (pF) to microfarad (μF) range. They are described by the dielectric material they are made of and by whether they are of fixed or variable type. Figure 6.3 shows the circuit symbols for fixed and variable capacitors. Note that according to the passive sign convention, current is considered to flow into the positive terminal of the capacitor when the capacitor is being charged, and out of the positive terminal when the capacitor is discharging.

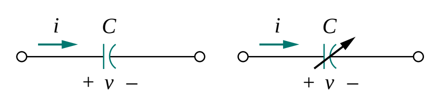


Figure 6.3 Circuit symbols for capacitors: (a) fixed capacitor, (b) variable capacitor.

Figure 6.4 shows common types of fixed-value capacitors. Polyester capacitors are light in weight, stable, and their change with temperature is predictable. Instead of polyester, other dielectric materials such as mica and polystyrene may be used. Film capacitors are rolled and housed in metal or plastic films. Electrolytic capacitors produce very high capacitance. Figure 6.5 shows the most common types of variable capacitors. The capacitance of a trimmer (or padder) capacitor or a glass piston capacitor is varied by turning the screw. The trimmer capacitor is often placed in parallel with another capacitor so that the equivalent capacitance can be varied slightly. The capacitance of the variable air capacitor (meshed plates) is varied by turning the shaft. Variable capacitors are used in radio

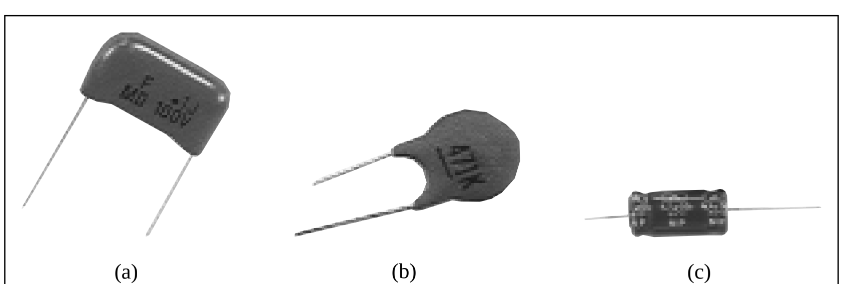


Figure 6.4 Fixed capacitors: (a) polyester capacitor, (b) ceramic capacitor, (c) electrolytic capacitor. *(Courtesy of Tech America.)*

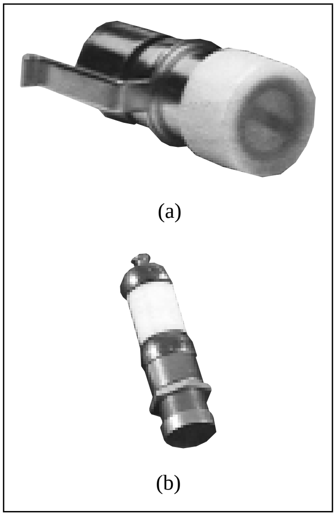


Figure 6.5 Variable capacitors: (a) trimmer capacitor, (b) filmtrim capacitor.
(Courtesy of Johanson.)

According to Eq. (6.4), for a capacitor to carry current, its voltage must vary with time. Hence, for constant voltage, $i = 0$.

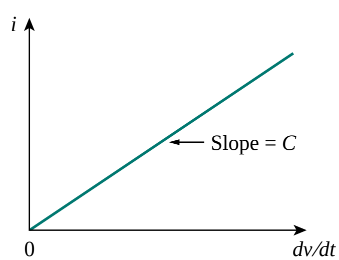


Figure 6.6 Current-voltage relationship of a capacitor.

receivers allowing one to tune to various stations. In addition, capacitors are used to block dc, pass ac, shift phase, store energy, start motors, and suppress noise.

To obtain the current-voltage relationship of the capacitor, we take the derivative of both sides of Eq. (6.1). Since

$$i = \frac{dq}{dt} \tag{6.3}$$

differentiating both sides of Eq. (6.1) gives

$$\boxed{i = C\frac{dv}{dt}} \tag{6.4}$$

This is the current-voltage relationship for a capacitor, assuming the positive sign convention. The relationship is illustrated in Fig. 6.6 for a capacitor whose capacitance is independent of voltage. Capacitors that satisfy Eq. (6.4) are said to be *linear.* For a *nonlinear capacitor*, the plot of the current-voltage relationship is not a straight line. Although some capacitors are nonlinear, most are linear. We will assume linear capacitors in this book.

The voltage-current relation of the capacitor can be obtained by integrating both sides of Eq. (6.4). We get

$$v = \frac{1}{C}\int_{-\infty}^{t} i\, dt \tag{6.5}$$

or

$$\boxed{v = \frac{1}{C}\int_{t_0}^{t} i\, dt + v(t_0)} \tag{6.6}$$

where $v(t_0) = q(t_0)/C$ is the voltage across the capacitor at time t_0. Equation (6.6) shows that capacitor voltage depends on the past history of the capacitor current. Hence, the capacitor has memory—a property that is often exploited.

The instantaneous power delivered to the capacitor is

$$p = vi = Cv\frac{dv}{dt} \tag{6.7}$$

The energy stored in the capacitor is therefore

$$w = \int_{-\infty}^{t} p\, dt = C\int_{-\infty}^{t} v\frac{dv}{dt}dt = C\int_{-\infty}^{t} v\, dv = \frac{1}{2}Cv^2\Big|_{t=-\infty}^{t} \tag{6.8}$$

We note that $v(-\infty) = 0$, because the capacitor was uncharged at $t = -\infty$. Thus,

$$\boxed{w = \frac{1}{2}Cv^2} \tag{6.9}$$

Using Eq. (6.1), we may rewrite Eq. (6.9) as

$$w = \frac{q^2}{2C} \tag{6.10}$$

Equation (6.9) or (6.10) represents the energy stored in the electric field that exists between the plates of the capacitor. This energy can be retrieved, since an ideal capacitor cannot dissipate energy. In fact, the word *capacitor* is derived from this element's capacity to store energy in an electric field.

We should note the following important properties of a capacitor:

1. Note from Eq. (6.4) that when the voltage across a capacitor is not changing with time (i.e., dc voltage), the current through the capacitor is zero. Thus,

> A capacitor is an open circuit to dc.

However, if a battery (dc voltage) is connected across a capacitor, the capacitor charges.

2. The voltage on the capacitor must be continuous.

> The voltage on a capacitor cannot change abruptly.

The capacitor resists an abrupt change in the voltage across it. According to Eq. (6.4), a discontinuous change in voltage requires an infinite current, which is physically impossible. For example, the voltage across a capacitor may take the form shown in Fig. 6.7(a), whereas it is not physically possible for the capacitor voltage to take the form shown in Fig. 6.7(b) because of the abrupt change. Conversely, the current through a capacitor can change instantaneously.

3. The ideal capacitor does not dissipate energy. It takes power from the circuit when storing energy in its field and returns previously stored energy when delivering power to the circuit.
4. A real, nonideal capacitor has a parallel-model leakage resistance, as shown in Fig. 6.8. The leakage resistance may be as high as 100 MΩ and can be neglected for most practical applications. For this reason, we will assume ideal capacitors in this book.

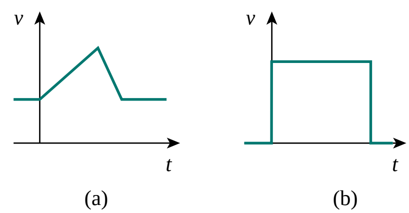


Figure 6.7 Voltage across a capacitor: (a) allowed, (b) not allowable; an abrupt change is not possible.

An alternative way of looking at this is using Eq. (6.9), which indicates that energy is proportional to voltage squared. Since injecting or extracting energy can only be done over some finite time, voltage cannot change instantaneously across a capacitor.

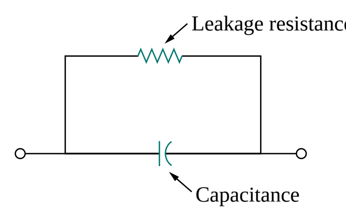


Figure 6.8 Circuit model of a nonideal capacitor.

EXAMPLE 6.1

(a) Calculate the charge stored on a 3-pF capacitor with 20 V across it.
(b) Find the energy stored in the capacitor.

Solution:

(a) Since $q = Cv$,

$$q = 3 \times 10^{-12} \times 20 = 60 \text{ pC}$$

(b) The energy stored is

$$w = \frac{1}{2}Cv^2 = \frac{1}{2} \times 3 \times 10^{-12} \times 400 = 600 \text{ pJ}$$

PRACTICE PROBLEM 6.1

What is the voltage across a 3-μF capacitor if the charge on one plate is 0.12 mC? How much energy is stored?

Answer: 40 V, 2.4 mJ.

EXAMPLE 6.2

The voltage across a 5-μF capacitor is

$$v(t) = 10 \cos 6000t \text{ V}$$

Calculate the current through it.

Solution:

By definition, the current is

$$\begin{aligned} i(t) &= C\frac{dv}{dt} = 5 \times 10^{-6}\frac{d}{dt}(10\cos 6000t) \\ &= -5 \times 10^{-6} \times 6000 \times 10 \sin 6000t = -0.3 \sin 6000t \text{ A} \end{aligned}$$

PRACTICE PROBLEM 6.2

If a 10-μF capacitor is connected to a voltage source with

$$v(t) = 50 \sin 2000t \text{ V}$$

determine the current through the capacitor.

Answer: $\cos 2000t$ A.

EXAMPLE 6.3

Determine the voltage across a 2-μF capacitor if the current through it is

$$i(t) = 6e^{-3000t} \text{ mA}$$

Assume that the initial capacitor voltage is zero.

Solution:

Since $v = \dfrac{1}{C}\displaystyle\int_0^t i\, dt + v(0)$ and $v(0) = 0$,

$$\begin{aligned} v &= \frac{1}{2 \times 10^{-6}}\int_0^t 6e^{-3000t}\, dt \cdot 10^{-3} \\ &= \frac{3 \times 10^3}{-3000} e^{-3000t}\Big|_0^t = (1 - e^{-3000t}) \text{ V} \end{aligned}$$

PRACTICE PROBLEM 6.3

The current through a 100-μF capacitor is $i(t) = 50 \sin 120\pi t$ mA. Calculate the voltage across it at $t = 1$ ms and $t = 5$ ms. Take $v(0) = 0$.

Answer: -93.137 V, -1.736 V.

EXAMPLE 6.4

Determine the current through a 200-μF capacitor whose voltage is shown in Fig. 6.9.

Solution:

The voltage waveform can be described mathematically as

$$v(t) = \begin{cases} 50t \text{ V} & 0 < t < 1 \\ 100 - 50t \text{ V} & 1 < t < 3 \\ -200 + 50t \text{ V} & 3 < t < 4 \\ 0 & \text{otherwise} \end{cases}$$

Since $i = C\, dv/dt$ and $C = 200\ \mu$F, we take the derivative of v to obtain

$$i(t) = 200 \times 10^{-6} \times \begin{cases} 50 & 0 < t < 1 \\ -50 & 1 < t < 3 \\ 50 & 3 < t < 4 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 10 \text{ mA} & 0 < t < 1 \\ -10 \text{ mA} & 1 < t < 3 \\ 10 \text{ mA} & 3 < t < 4 \\ 0 & \text{otherwise} \end{cases}$$

Thus the current waveform is as shown in Fig. 6.10.

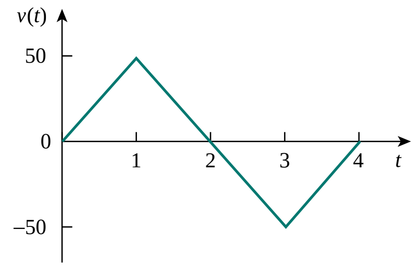


Figure 6.9 For Example 6.4.

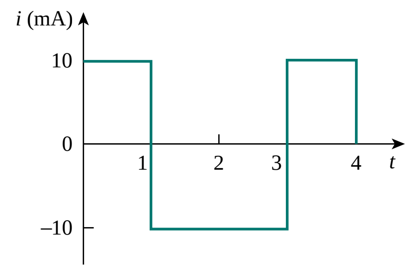


Figure 6.10 For Example 6.4.

PRACTICE PROBLEM 6.4

An initially uncharged 1-mF capacitor has the current shown in Fig. 6.11 across it. Calculate the voltage across it at $t = 2$ ms and $t = 5$ ms.

Answer: 100 mV, 400 mV.

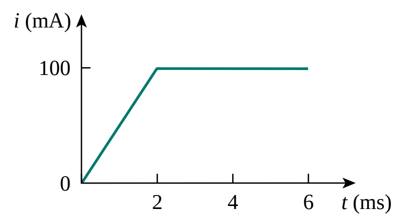


Figure 6.11 For Practice Prob. 6.4.

EXAMPLE 6.5

Obtain the energy stored in each capacitor in Fig. 6.12(a) under dc conditions.

Solution:

Under dc conditions, we replace each capacitor with an open circuit, as shown in Fig. 6.12(b). The current through the series combination of the 2-kΩ and 4-kΩ resistors is obtained by current division as

$$i = \frac{3}{3+2+4}(6 \text{ mA}) = 2 \text{ mA}$$

Hence, the voltages v_1 and v_2 across the capacitors are

$$v_1 = 2000i = 4 \text{ V} \qquad v_2 = 4000i = 8 \text{ V}$$

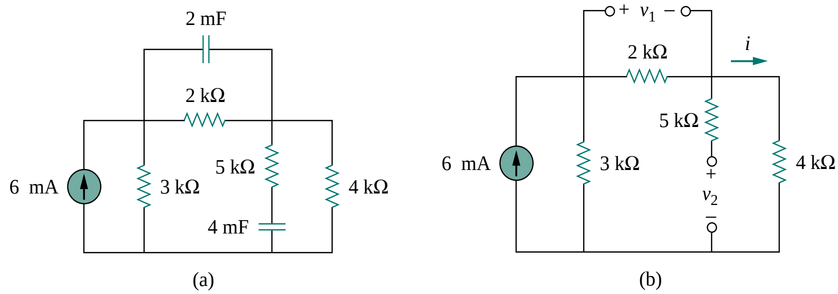


Figure 6.12 For Example 6.5.

and the energies stored in them are

$$w_1 = \frac{1}{2}C_1 v_1^2 = \frac{1}{2}(2 \times 10^{-3})(4)^2 = 16 \text{ mJ}$$

$$w_2 = \frac{1}{2}C_2 v_2^2 = \frac{1}{2}(4 \times 10^{-3})(8)^2 = 128 \text{ mJ}$$

PRACTICE PROBLEM 6.5

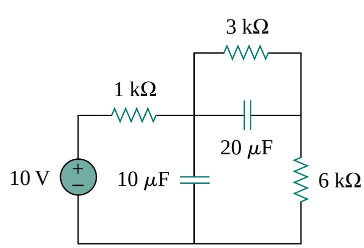


Figure 6.13 For Practice Prob. 6.5.

Under dc conditions, find the energy stored in the capacitors in Fig. 6.13.

Answer: 405 μJ, 90 μJ.

6.3 SERIES AND PARALLEL CAPACITORS

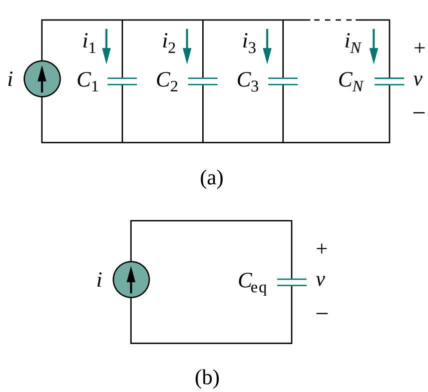


Figure 6.14 (a) Parallel-connected N capacitors, (b) equivalent circuit for the parallel capacitors.

We know from resistive circuits that series-parallel combination is a powerful tool for reducing circuits. This technique can be extended to series-parallel connections of capacitors, which are sometimes encountered. We desire to replace these capacitors by a single equivalent capacitor C_{eq}.

In order to obtain the equivalent capacitor C_{eq} of N capacitors in parallel, consider the circuit in Fig. 6.14(a). The equivalent circuit is in Fig. 6.14(b). Note that the capacitors have the same voltage v across them. Applying KCL to Fig. 6.14(a),

$$i = i_1 + i_2 + i_3 + \cdots + i_N \tag{6.11}$$

But $i_k = C_k \, dv/dt$. Hence,

$$\begin{aligned} i &= C_1\frac{dv}{dt} + C_2\frac{dv}{dt} + C_3\frac{dv}{dt} + \cdots + C_N\frac{dv}{dt} \\ &= \left(\sum_{k=1}^{N} C_k\right)\frac{dv}{dt} = C_{eq}\frac{dv}{dt} \end{aligned} \tag{6.12}$$

where

$$\boxed{C_{eq} = C_1 + C_2 + C_3 + \cdots + C_N} \tag{6.13}$$

The equivalent capacitance of N parallel-connected capacitors is the sum of the individual capacitances.

We observe that capacitors in parallel combine in the same manner as resistors in series.

We now obtain C_{eq} of N capacitors connected in series by comparing the circuit in Fig. 6.15(a) with the equivalent circuit in Fig. 6.15(b). Note that the same current i flows (and consequently the same charge) through the capacitors. Applying KVL to the loop in Fig. 6.15(a),

$$v = v_1 + v_2 + v_3 + \cdots + v_N \tag{6.14}$$

But $v_k = \dfrac{1}{C_k}\displaystyle\int_{t_0}^{t} i(t)\,dt + v_k(t_0)$. Therefore,

$$\begin{aligned} v &= \frac{1}{C_1}\int_{t_0}^{t} i(t)\,dt + v_1(t_0) + \frac{1}{C_2}\int_{t_0}^{t} i(t)\,dt + v_2(t_0) \\ &\qquad + \cdots + \frac{1}{C_N}\int_{t_0}^{t} i(t)\,dt + v_N(t_0) \\ &= \left(\frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N}\right)\int_{t_0}^{t} i(t)\,dt + v_1(t_0) + v_2(t_0) \\ &\qquad + \cdots + v_N(t_0) \\ &= \frac{1}{C_{eq}}\int_{t_0}^{t} i(t)\,dt + v(t_0) \end{aligned} \tag{6.15}$$

where

$$\boxed{\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots + \frac{1}{C_N}} \tag{6.16}$$

The initial voltage $v(t_0)$ across C_{eq} is required by KVL to be the sum of the capacitor voltages at t_0. Or according to Eq. (6.15),

$$v(t_0) = v_1(t_0) + v_2(t_0) + \cdots + v_N(t_0)$$

Thus, according to Eq. (6.16),

The equivalent capacitance of series-connected capacitors is the reciprocal of the sum of the reciprocals of the individual capacitances.

Note that capacitors in series combine in the same manner as resistors in parallel. For $N = 2$ (i.e., two capacitors in series), Eq. (6.16) becomes

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$

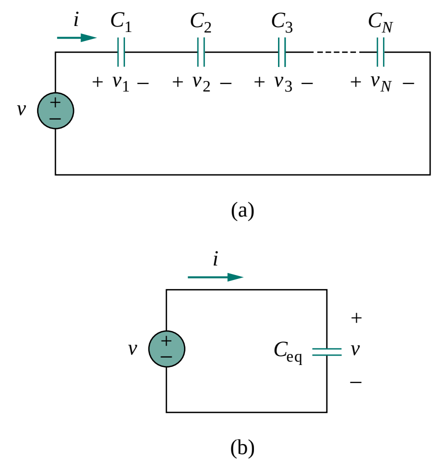


Figure 6.15 (a) Series-connected N capacitors, (b) equivalent circuit for the series capacitor.

or

$$\boxed{C_{eq} = \frac{C_1 C_2}{C_1 + C_2}} \tag{6.17}$$

EXAMPLE 6.6

Find the equivalent capacitance seen between terminals a and b of the circuit in Fig. 6.16.

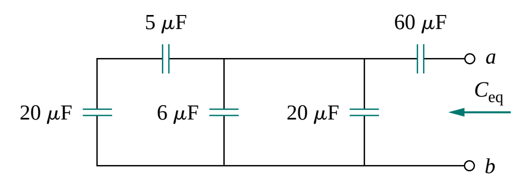


Figure 6.16 For Example 6.6.

Solution:

The 20-μF and 5-μF capacitors are in series; their equivalent capacitance is

$$\frac{20 \times 5}{20 + 5} = 4\ \mu\text{F}$$

This 4-μF capacitor is in parallel with the 6-μF and 20-μF capacitors; their combined capacitance is

$$4 + 6 + 20 = 30\ \mu\text{F}$$

This 30-μF capacitor is in series with the 60-μF capacitor. Hence, the equivalent capacitance for the entire circuit is

$$C_{eq} = \frac{30 \times 60}{30 + 60} = 20\ \mu\text{F}$$

PRACTICE PROBLEM 6.6

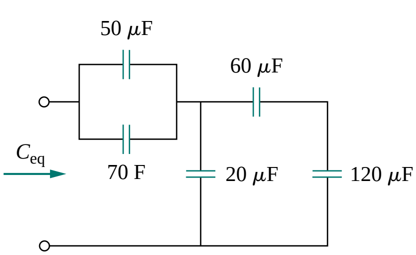


Figure 6.17 For Practice Prob. 6.6.

Find the equivalent capacitance seen at the terminals of the circuit in Fig. 6.17.

Answer: 40 μF.

EXAMPLE 6.7

For the circuit in Fig. 6.18, find the voltage across each capacitor.

Solution:

We first find the equivalent capacitance C_{eq}, shown in Fig. 6.19. The two parallel capacitors in Fig. 6.18 can be combined to get $40 + 20 = 60$ mF. This 60-mF capacitor is in series with the 20-mF and 30-mF capacitors. Thus,

$$C_{eq} = \frac{1}{\frac{1}{60} + \frac{1}{30} + \frac{1}{20}} \text{ mF} = 10 \text{ mF}$$

The total charge is

$$q = C_{eq}v = 10 \times 10^{-3} \times 30 = 0.3 \text{ C}$$

This is the charge on the 20-mF and 30-mF capacitors, because they are in series with the 30-V source. (A crude way to see this is to imagine that charge acts like current, since $i = dq/dt$.) Therefore,

$$v_1 = \frac{q}{C_1} = \frac{0.3}{20 \times 10^{-3}} = 15 \text{ V} \qquad v_2 = \frac{q}{C_2} = \frac{0.3}{30 \times 10^{-3}} = 10 \text{ V}$$

Having determined v_1 and v_2, we now use KVL to determine v_3 by

$$v_3 = 30 - v_1 - v_2 = 5 \text{ V}$$

Alternatively, since the 40-mF and 20-mF capacitors are in parallel, they have the same voltage v_3 and their combined capacitance is $40 + 20 = 60$ mF. This combined capacitance is in series with the 20-mF and 30-mF capacitors and consequently has the same charge on it. Hence,

$$v_3 = \frac{q}{60 \text{ mF}} = \frac{0.3}{60 \times 10^{-3}} = 5 \text{ V}$$

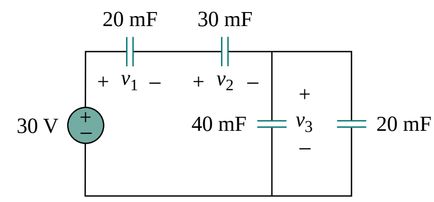


Figure 6.18 For Example 6.7.

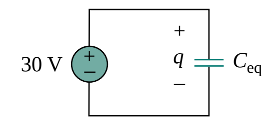


Figure 6.19 Equivalent circuit for Fig. 6.18.

PRACTICE PROBLEM 6.7

Find the voltage across each of the capacitors in Fig. 6.20.

Answer: $v_1 = 30$ V, $v_2 = 30$ V, $v_3 = 10$ V, $v_4 = 20$ V.

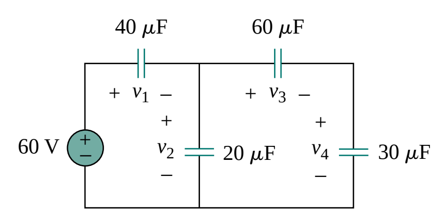


Figure 6.20 For Practice Prob. 6.7.

6.4 INDUCTORS

An inductor is a passive element designed to store energy in its magnetic field. Inductors find numerous applications in electronic and power systems. They are used in power supplies, transformers, radios, TVs, radars, and electric motors.

Any conductor of electric current has inductive properties and may be regarded as an inductor. But in order to enhance the inductive effect, a practical inductor is usually formed into a cylindrical coil with many turns of conducting wire, as shown in Fig. 6.21.

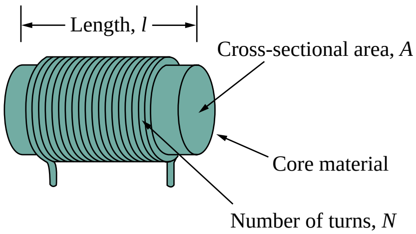


Figure 6.21 Typical form of an inductor.

An **inductor** consists of a coil of conducting wire.

If current is allowed to pass through an inductor, it is found that the voltage across the inductor is directly proportional to the time rate of change of the current. Using the passive sign convention,

$$\boxed{v = L\frac{di}{dt}} \tag{6.18}$$

where L is the constant of proportionality called the *inductance* of the inductor. The unit of inductance is the henry (H), named in honor of the American inventor Joseph Henry (1797–1878). It is clear from Eq. (6.18) that 1 henry equals 1 volt-second per ampere.

In view of Eq. (6.18), for an inductor to have voltage across its terminals, its current must vary with time. Hence, $v = 0$ for constant current through the inductor.

Inductance is the property whereby an inductor exhibits opposition to the change of current flowing through it, measured in henrys (H).

The inductance of an inductor depends on its physical dimension and construction. Formulas for calculating the inductance of inductors of different shapes are derived from electromagnetic theory and can be found in standard electrical engineering handbooks. For example, for the inductor (solenoid) shown in Fig. 6.21,

$$L = \frac{N^2\mu A}{\ell} \tag{6.19}$$

where N is the number of turns, ℓ is the length, A is the cross-sectional area, and μ is the permeability of the core. We can see from Eq. (6.19) that inductance can be increased by increasing the number of turns of coil, using material with higher permeability as the core, increasing the cross-sectional area, or reducing the length of the coil.

Like capacitors, commercially available inductors come in different values and types. Typical practical inductors have inductance values ranging from a few microhenrys (μH), as in communication systems, to tens of henrys (H) as in power systems. Inductors may be fixed or variable. The core may be made of iron, steel, plastic, or air. The terms *coil* and *choke* are also used for inductors. Common inductors are shown in Fig. 6.22. The circuit symbols for inductors are shown in Fig. 6.23, following the passive sign convention.

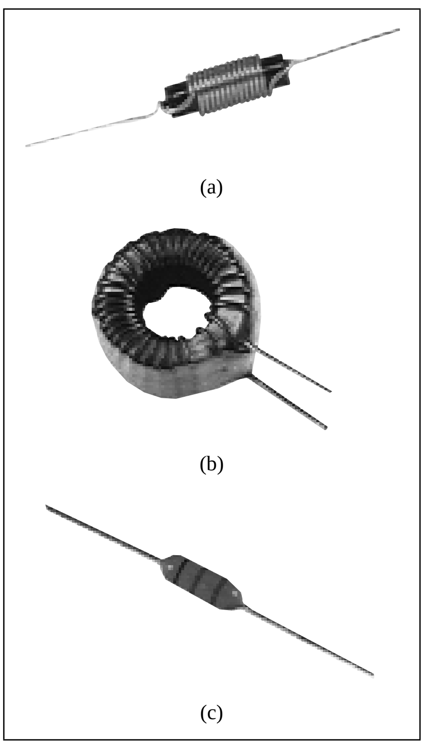


Figure 6.22 Various types of inductors: (a) solenoidal wound inductor, (b) toroidal inductor, (c) chip inductor. *(Courtesy of Tech America.)*

Equation (6.18) is the voltage-current relationship for an inductor. Figure 6.24 shows this relationship graphically for an inductor whose inductance is independent of current. Such an inductor is known as a *linear inductor*. For a *nonlinear inductor*, the plot of Eq. (6.18) will not be a straight line because its inductance varies with current. We will assume linear inductors in this textbook unless stated otherwise.

The current-voltage relationship is obtained from Eq. (6.18) as

$$di = \frac{1}{L}v\,dt$$

Integrating gives

$$i = \frac{1}{L}\int_{-\infty}^{t} v(t)\,dt \tag{6.20}$$

or

$$\boxed{i = \frac{1}{L}\int_{t_0}^{t} v(t)\,dt + i(t_0)} \tag{6.21}$$

where $i(t_0)$ is the total current for $-\infty < t < t_0$ and $i(-\infty) = 0$. The idea of making $i(-\infty) = 0$ is practical and reasonable, because there must be a time in the past when there was no current in the inductor.

The inductor is designed to store energy in its magnetic field. The energy stored can be obtained from Eqs. (6.18) and (6.20). The power delivered to the inductor is

$$p = vi = \left(L\frac{di}{dt}\right)i \tag{6.22}$$

The energy stored is

$$\begin{aligned} w &= \int_{-\infty}^{t} p\,dt = \int_{-\infty}^{t} \left(L\frac{di}{dt}\right) i\,dt \\ &= L\int_{-\infty}^{t} i\,di = \frac{1}{2}Li^2(t) - \frac{1}{2}Li^2(-\infty) \end{aligned} \tag{6.23}$$

Since $i(-\infty) = 0$,

$$\boxed{w = \frac{1}{2}Li^2} \tag{6.24}$$

We should note the following important properties of an inductor.

1. Note from Eq. (6.18) that the voltage across an inductor is zero when the current is constant. Thus,

An inductor acts like a short circuit to dc.

2. An important property of the inductor is its opposition to the change in current flowing through it.

The current through an inductor cannot change instantaneously.

According to Eq. (6.18), a discontinuous change in the current through an inductor requires an infinite voltage, which is not physically possible. Thus, an inductor opposes an abrupt change in the current through it. For example, the current through an inductor may take the form shown in Fig. 6.25(a), whereas the inductor current cannot take the form shown in Fig. 6.25(b) in real-life situations due to the discontinuities. However, the voltage across an inductor can change abruptly.

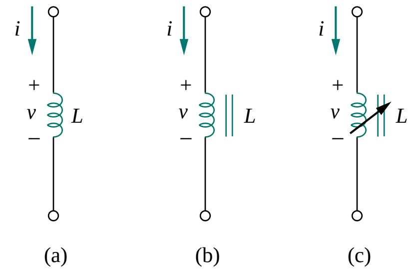


Figure 6.23 Circuit symbols for inductors: (a) air-core, (b) iron-core, (c) variable iron-core.

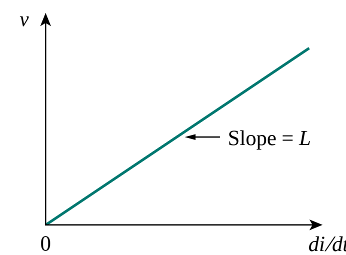


Figure 6.24 Voltage-current relationship of an inductor.

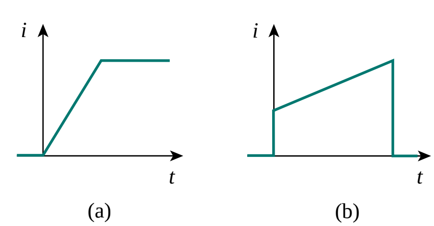


Figure 6.25 Current through an inductor: (a) allowed, (b) not allowable; an abrupt change is not possible.

3. Like the ideal capacitor, the ideal inductor does not dissipate energy. The energy stored in it can be retrieved at a later time. The inductor takes power from the circuit when storing energy and delivers power to the circuit when returning previously stored energy.

Since an inductor is often made of a highly conducting wire, it has a very small resistance.

4. A practical, nonideal inductor has a significant resistive component, as shown in Fig. 6.26. This is due to the fact that the inductor is made of a conducting material such as copper, which has some resistance. This resistance is called the *winding resistance* R_w, and it appears in series with the inductance of the inductor. The presence of R_w makes it both an energy storage device and an energy dissipation device. Since R_w is usually very small, it is ignored in most cases. The nonideal inductor also has a *winding capacitance* C_w due to the capacitive coupling between the conducting coils. C_w is very small and can be ignored in most cases, except at high frequencies. We will assume ideal inductors in this book.

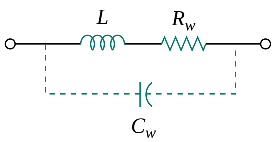


Figure 6.26 Circuit model for a practical inductor.

EXAMPLE 6.8

The current through a 0.1-H inductor is $i(t) = 10te^{-5t}$ A. Find the voltage across the inductor and the energy stored in it.

Solution:

Since $v = L\,di/dt$ and $L = 0.1$ H,

$$v = 0.1\frac{d}{dt}(10te^{-5t}) = e^{-5t} + t(-5)e^{-5t} = e^{-5t}(1-5t)\text{ V}$$

The energy stored is

$$w = \frac{1}{2}Li^2 = \frac{1}{2}(0.1)100t^2e^{-10t} = 5t^2e^{-10t}\text{ J}$$

PRACTICE PROBLEM 6.8

If the current through a 1-mH inductor is $i(t) = 20\cos 100t$ mA, find the terminal voltage and the energy stored.

Answer: $-2\sin 100t$ mV, $0.2\cos^2 100t\ \mu$J.

EXAMPLE 6.9

Find the current through a 5-H inductor if the voltage across it is

$$v(t) = \begin{cases} 30t^2, & t > 0 \\ 0, & t < 0 \end{cases}$$

Also find the energy stored within $0 < t < 5$ s.

Solution:

Since $i = \dfrac{1}{L}\displaystyle\int_{t_0}^{t} v(t)\,dt + i(t_0)$ and $L = 5$ H,

$$i = \frac{1}{5}\int_0^t 30t^2\,dt + 0 = 6 \times \frac{t^3}{3} = 2t^3\text{ A}$$

The power $p = vi = 60t^5$, and the energy stored is then

$$w = \int p\, dt = \int_0^5 60t^5\, dt = 60\frac{t^6}{6}\bigg|_0^5 = 156.25 \text{ kJ}$$

Alternatively, we can obtain the energy stored using Eq. (6.13), by writing

$$w\big|_0^5 = \frac{1}{2}Li^2(5) - \frac{1}{2}Li(0) = \frac{1}{2}(5)(2 \times 5^3)^2 - 0 = 156.25 \text{ kJ}$$

as obtained before.

PRACTICE PROBLEM 6.9

The terminal voltage of a 2-H inductor is $v = 10(1-t)$ V. Find the current flowing through it at $t = 4$ s and the energy stored in it within $0 < t < 4$ s. Assume $i(0) = 2$ A.

Answer: -18 A, 320 J.

EXAMPLE 6.10

Consider the circuit in Fig. 6.27(a). Under dc conditions, find: (a) i, v_C, and i_L, (b) the energy stored in the capacitor and inductor.

Solution:

(a) Under dc conditions, we replace the capacitor with an open circuit and the inductor with a short circuit, as in Fig. 6.27(b). It is evident from Fig. 6.27(b) that

$$i = i_L = \frac{12}{1+5} = 2 \text{ A}$$

The voltage v_C is the same as the voltage across the 5-Ω resistor. Hence,

$$v_C = 5i = 10 \text{ V}$$

(b) The energy in the capacitor is

$$w_C = \frac{1}{2}Cv_C^2 = \frac{1}{2}(1)(10^2) = 50 \text{ J}$$

and that in the inductor is

$$w_L = \frac{1}{2}Li_L^2 = \frac{1}{2}(2)(2^2) = 4 \text{ J}$$

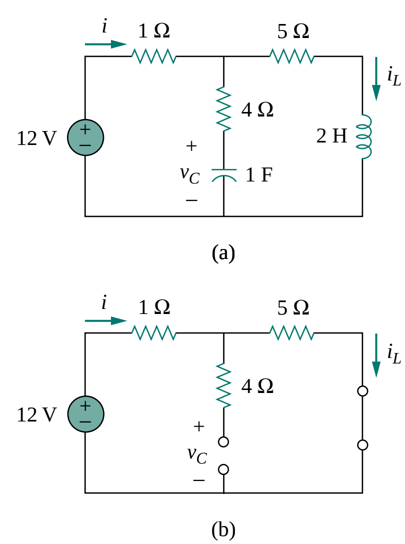


Figure 6.27 For Example 6.10.

PRACTICE PROBLEM 6.10

Determine v_C, i_L, and the energy stored in the capacitor and inductor in the circuit of Fig. 6.28 under dc conditions.

Answer: 3 V, 3 A, 9 J, 1.125 J.

@ Network Analysis

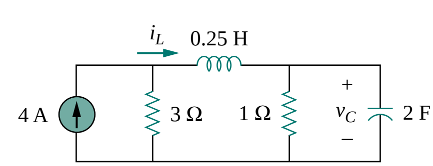


Figure 6.28 For Practice Prob. 6.10.

6.5 SERIES AND PARALLEL INDUCTORS

Now that the inductor has been added to our list of passive elements, it is necessary to extend the powerful tool of series-parallel combination. We need to know how to find the equivalent inductance of a series-connected or parallel-connected set of inductors found in practical circuits.

Consider a series connection of N inductors, as shown in Fig. 6.29(a), with the equivalent circuit shown in Fig. 6.29(b). The inductors have the same current through them. Applying KVL to the loop,

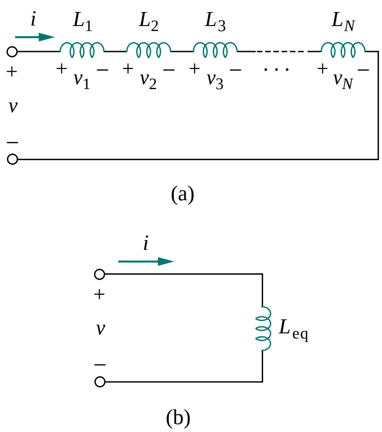


Figure 6.29 (a) A series connection of N inductors, (b) equivalent circuit for the series inductors.

$$v = v_1 + v_2 + v_3 + \cdots + v_N \tag{6.25}$$

Substituting $v_k = L_k\, di/dt$ results in

$$\begin{aligned} v &= L_1\frac{di}{dt} + L_2\frac{di}{dt} + L_3\frac{di}{dt} + \cdots + L_N\frac{di}{dt} \\ &= (L_1 + L_2 + L_3 + \cdots + L_N)\frac{di}{dt} \\ &= \left(\sum_{k=1}^{N} L_k\right)\frac{di}{dt} = L_{\text{eq}}\frac{di}{dt} \end{aligned} \tag{6.26}$$

where

$$\boxed{L_{\text{eq}} = L_1 + L_2 + L_3 + \cdots + L_N} \tag{6.27}$$

Thus,

The equivalent inductance of series-connected inductors is the sum of the individual inductances.

Inductors in series are combined in exactly the same way as resistors in series.

We now consider a parallel connection of N inductors, as shown in Fig. 6.30(a), with the equivalent circuit in Fig. 6.30(b). The inductors have the same voltage across them. Using KCL,

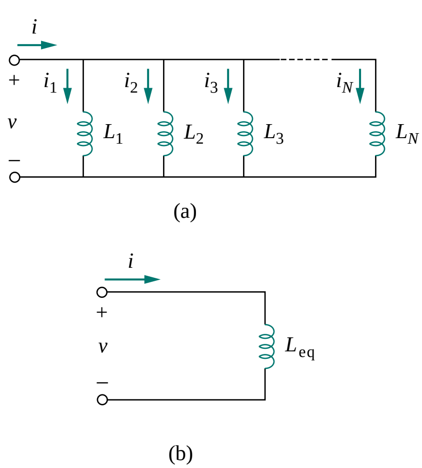


Figure 6.30 (a) A parallel connection of N inductors, (b) equivalent circuit for the parallel inductors.

$$i = i_1 + i_2 + i_3 + \cdots + i_N \tag{6.28}$$

But $i_k = \dfrac{1}{L_k}\displaystyle\int_{t_0}^{t} v\, dt + i_k(t_0)$; hence,

$$\begin{aligned} i &= \frac{1}{L_1}\int_{t_0}^{t} v\, dt + i_1(t_0) + \frac{1}{L_2}\int_{t_0}^{t} v\, dt + i_2(t_0) \\ &\quad + \cdots + \frac{1}{L_N}\int_{t_0}^{t} v\, dt + i_N(t_0) \\ &= \left(\frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_N}\right)\int_{t_0}^{t} v\, dt + i_1(t_0) + i_2(t_0) \\ &\quad + \cdots + i_N(t_0) \\ &= \left(\sum_{k=1}^{N}\frac{1}{L_k}\right)\int_{t_0}^{t} v\, dt + \sum_{k=1}^{N} i_k(t_0) = \frac{1}{L_{\text{eq}}}\int_{t_0}^{t} v\, dt + i(t_0) \end{aligned} \tag{6.29}$$

where

$$\boxed{\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \cdots + \frac{1}{L_N}} \tag{6.30}$$

The initial current $i(t_0)$ through L_{eq} at $t = t_0$ is expected by KCL to be the sum of the inductor currents at t_0. Thus, according to Eq. (6.29),

$$i(t_0) = i_1(t_0) + i_2(t_0) + \cdots + i_N(t_0)$$

According to Eq. (6.30),

> The **equivalent inductance** of parallel inductors is the reciprocal of the sum of the reciprocals of the individual inductances.

Note that the inductors in parallel are combined in the same way as resistors in parallel.

For two inductors in parallel ($N = 2$), Eq. (6.30) becomes

$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} \quad \text{or} \quad L_{eq} = \frac{L_1 L_2}{L_1 + L_2} \tag{6.31}$$

It is appropriate at this point to summarize the most important characteristics of the three basic circuit elements we have studied. The summary is given in Table 6.1.

TABLE 6.1 Important characteristics of the basic elements.†

Relation	Resistor (R)	Capacitor (C)	Inductor (L)
v-i:	$v = iR$	$v = \frac{1}{C}\int_{t_0}^{t} i\,dt + v(t_0)$	$v = L\frac{di}{dt}$
i-v:	$i = v/R$	$i = C\frac{dv}{dt}$	$i = \frac{1}{L}\int_{t_0}^{t} i\,dt + i(t_0)$
p or w:	$p = i^2 R = \frac{v^2}{R}$	$w = \frac{1}{2}Cv^2$	$w = \frac{1}{2}Li^2$
Series:	$R_{eq} = R_1 + R_2$	$C_{eq} = \frac{C_1 C_2}{C_1 + C_2}$	$L_{eq} = L_1 + L_2$
Parallel:	$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$	$C_{eq} = C_1 + C_2$	$L_{eq} = \frac{L_1 L_2}{L_1 + L_2}$
At dc:	Same	Open circuit	Short circuit
Circuit variable that cannot change abruptly:	Not applicable	v	i

† *Passive sign convention is assumed.*

EXAMPLE 6.11

Find the equivalent inductance of the circuit shown in Fig. 6.31.

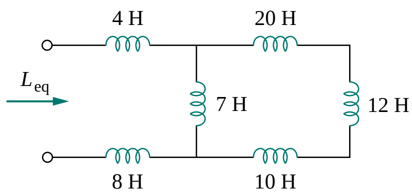


Figure 6.31 For Example 6.11.

Solution:

The 10-H, 12-H, and 20-H inductors are in series; thus, combining them gives a 42-H inductance. This 42-H inductor is in parallel with the 7-H inductor so that they are combined, to give

$$\frac{7 \times 42}{7 + 42} = 6 \text{ H}$$

This 6-H inductor is in series with the 4-H and 8-H inductors. Hence,

$$L_{eq} = 4 + 6 + 8 = 18 \text{ H}$$

PRACTICE PROBLEM 6.11

Calculate the equivalent inductance for the inductive ladder network in Fig. 6.32.

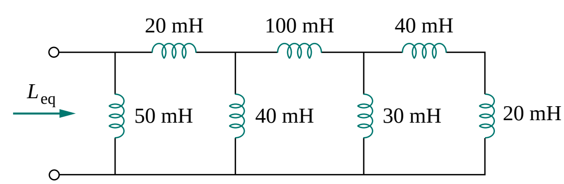


Figure 6.32 For Practice Prob. 6.11.

Answer: 25 mH.

EXAMPLE 6.12

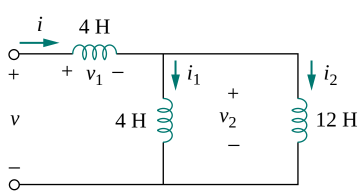


Figure 6.33 For Example 6.12.

For the circuit in Fig. 6.33, $i(t) = 4(2 - e^{-10t})$ mA. If $i_2(0) = -1$ mA, find: (a) $i_1(0)$; (b) $v(t)$, $v_1(t)$, and $v_2(t)$; (c) $i_1(t)$ and $i_2(t)$.

Solution:

(a) From $i(t) = 4(2 - e^{-10t})$ mA, $i(0) = 4(2 - 1) = 4$ mA. Since $i = i_1 + i_2$,

$$i_1(0) = i(0) - i_2(0) = 4 - (-1) = 5 \text{ mA}$$

(b) The equivalent inductance is

$$L_{eq} = 2 + 4 \parallel 12 = 2 + 3 = 5 \text{ H}$$

Thus,

$$v(t) = L_{eq}\frac{di}{dt} = 5(4)(-1)(-10)e^{-10t} \text{ mV} = 200e^{-10t} \text{ mV}$$

and

$$v_1(t) = 2\frac{di}{dt} = 2(-4)(-10)e^{-10t} \text{ mV} = 80e^{-10t} \text{ mV}$$

Since $v = v_1 + v_2$,

$$v_2(t) = v(t) - v_1(t) = 120e^{-10t} \text{ mV}$$

(c) The current i_1 is obtained as

$$i_1(t) = \frac{1}{4}\int_0^t v_2\,dt + i_1(0) = \frac{120}{4}\int_0^t e^{-10t}\,dt + 5\text{ mA}$$
$$= -3e^{-10t}\Big|_0^t + 5\text{ mA} = -3e^{-10t} + 3 + 5 = 8 - 3e^{-10t}\text{ mA}$$

Similarly,

$$i_2(t) = \frac{1}{12}\int_0^t v_2\,dt + i_2(0) = \frac{120}{12}\int_0^t e^{-10t}\,dt - 1\text{ mA}$$
$$= -e^{-10t}\Big|_0^t - 1\text{ mA} = -e^{-10t} + 1 - 1 = -e^{-10t}\text{ mA}$$

Note that $i_1(t) + i_2(t) = i(t)$.

PRACTICE PROBLEM 6.12

In the circuit of Fig. 6.34, $i_1(t) = 0.6e^{-2t}$ A. If $i(0) = 1.4$ A, find: (a) $i_2(0)$; (b) $i_2(t)$ and $i(t)$; (c) $v(t)$, $v_1(t)$, and $v_2(t)$.

Answer: (a) 0.8 A, (b) $(-0.4 + 1.2e^{-2t})$ A, $(-0.4 + 1.8e^{-2t})$ A, (c) $-7.2e^{-2t}$ V, $-28.8e^{-2t}$ V, $-36e^{-2t}$ V.

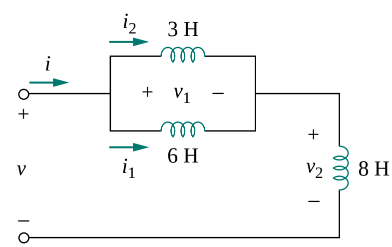


Figure 6.34 For Practice Prob. 6.12.

†6.6 APPLICATIONS

Circuit elements such as resistors and capacitors are commercially available in either discrete form or integrated-circuit (IC) form. Unlike capacitors and resistors, inductors with appreciable inductance are difficult to produce on IC substrates. Therefore, inductors (coils) usually come in discrete form and tend to be more bulky and expensive. For this reason, inductors are not as versatile as capacitors and resistors, and they are more limited in applications. However, there are several applications in which inductors have no practical substitute. They are routinely used in relays, delays, sensing devices, pick-up heads, telephone circuits, radio and TV receivers, power supplies, electric motors, microphones, and loudspeakers, to mention a few.

Capacitors and inductors possess the following three special properties that make them very useful in electric circuits:

1. The capacity to store energy makes them useful as temporary voltage or current sources. Thus, they can be used for generating a large amount of current or voltage for a short period of time.
2. Capacitors oppose any abrupt change in voltage, while inductors oppose any abrupt change in current. This property

makes inductors useful for spark or arc suppression and for converting pulsating dc voltage into relatively smooth dc voltage.

3. Capacitors and inductors are frequency sensitive. This property makes them useful for frequency discrimination.

The first two properties are put to use in dc circuits, while the third one is taken advantage of in ac circuits. We will see how useful these properties are in later chapters. For now, consider three applications involving capacitors and op amps: integrator, differentiator, and analog computer.

6.6.1 Integrator

Important op amp circuits that use energy-storage elements include integrators and differentiators. These op amp circuits often involve resistors and capacitors; inductors (coils) tend to be more bulky and expensive.

The op amp integrator is used in numerous applications, especially in analog computers, to be discussed in Section 6.6.3.

> An **integrator** is an op amp circuit whose output is proportional to the integral of the input signal.

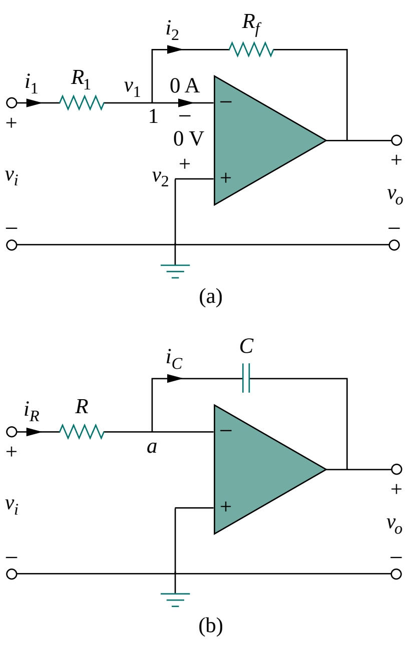


Figure 6.35 Replacing the feedback resistor in the inverting amplifier in (a) produces an integrator in (b).

If the feedback resistor R_f in the familiar inverting amplifier of Fig. 6.35(a) is replaced by a capacitor, we obtain an ideal integrator, as shown in Fig. 6.35(b). It is interesting that we can obtain a mathematical representation of integration this way. At node a in Fig. 6.35(b),

$$i_R = i_C \tag{6.32}$$

But

$$i_R = \frac{v_i}{R}, \qquad i_C = -C\frac{dv_o}{dt}$$

Substituting these in Eq. (6.32), we obtain

$$\frac{v_i}{R} = -C\frac{dv_o}{dt} \tag{6.33a}$$

$$dv_o = -\frac{1}{RC}v_i\,dt \tag{6.33b}$$

Integrating both sides gives

$$v_o(t) - v_o(0) = -\frac{1}{RC}\int_0^t v_i(t)\,dt \tag{6.34}$$

To ensure that $v_o(0) = 0$, it is always necessary to discharge the integrator's capacitor prior to the application of a signal. Assuming $v_o(0) = 0$,

$$\boxed{v_o = -\frac{1}{RC}\int_0^t v_i(t)\,dt} \tag{6.35}$$

which shows that the circuit in Fig. 6.35(b) provides an output voltage proportional to the integral of the input. In practice, the op amp integrator requires a feedback resistor to reduce dc gain and prevent saturation. Care must be taken that the op amp operates within the linear range so that it does not saturate.

EXAMPLE 6.13

If $v_1 = 10\cos 2t$ mV and $v_2 = 0.5t$ mV, find v_o in the op amp circuit in Fig. 6.36. Assume that the voltage across the capacitor is initially zero.

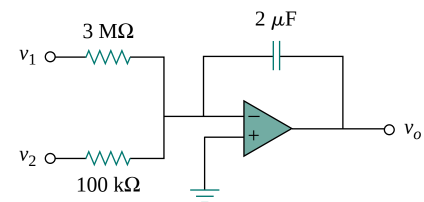


Figure 6.36 For Example 6.13.

Solution:

This is a summing integrator, and

$$\begin{aligned} v_o &= -\frac{1}{R_1 C}\int v_1\,dt - \frac{1}{R_2 C}\int v_2\,dt \\ &= -\frac{1}{3\times 10^6 \times 2\times 10^{-6}}\int_0^t 10\cos 2t\,dt \\ &\quad - \frac{1}{100\times 10^3\times 2\times 10^{-6}}\int_0^t 0.5t\,dt \\ &= -\frac{1}{6}\frac{10}{2}\sin 2t - \frac{1}{0.2}\frac{0.5t^2}{2} = -0.833\sin 2t - 1.25t^2 \text{ mV} \end{aligned}$$

PRACTICE PROBLEM 6.13

The integrator in Fig. 6.35 has $R = 25$ kΩ, $C = 10\ \mu$F. Determine the output voltage when a dc voltage of 10 mV is applied at $t = 0$. Assume that the op amp is initially nulled.

Answer: $-40t$ mV.

6.6.2 Differentiator

A **differentiator** is an op amp circuit whose output is proportional to the rate of change of the input signal.

In Fig. 6.35(a), if the input resistor is replaced by a capacitor, the resulting circuit is a differentiator, shown in Fig. 6.37. Applying KCL at node a,

$$i_R = i_C \tag{6.36}$$

But

$$i_R = -\frac{v_o}{R}, \qquad i_C = C\frac{dv_i}{dt}$$

Substituting these in Eq. (6.36) yields

$$\boxed{v_o = -RC\frac{dv_i}{dt}} \tag{6.37}$$

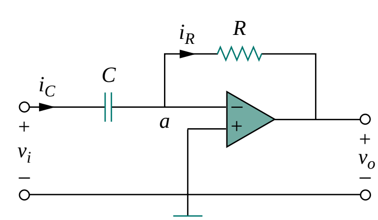


Figure 6.37 An op amp differentiator.

showing that the output is the derivative of the input. Differentiator circuits are electronically unstable because any electrical noise within the circuit is exaggerated by the differentiator. For this reason, the differentiator circuit in Fig. 6.37 is not as useful and popular as the integrator. It is seldom used in practice.

EXAMPLE 6.14

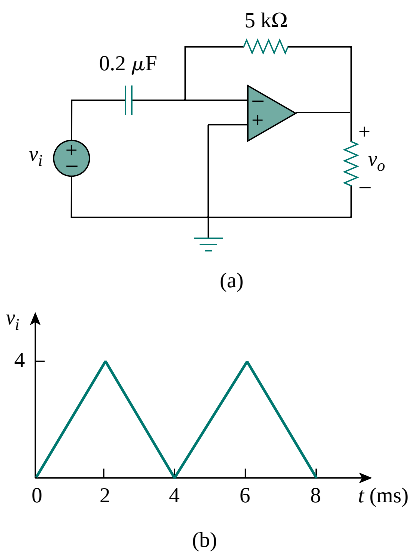


Figure 6.38 For Example 6.14.

Sketch the output voltage for the circuit in Fig. 6.38(a), given the input voltage in Fig. 6.38(b). Take $v_o = 0$ at $t = 0$.

Solution:

This is a differentiator with

$$RC = 5 \times 10^3 \times 0.2 \times 10^{-6} = 10^{-3} \text{ s}$$

For $0 < t < 4$ ms, we can express the input voltage in Fig. 6.38(b) as

$$v_i = \begin{cases} 2t & 0 < t < 2 \text{ ms} \\ 8 - 2t & 2 < t < 4 \text{ ms} \end{cases}$$

This is repeated for $4 < t < 8$. Using Eq. (6.37), the output is obtained as

$$v_o = -RC\frac{dv_i}{dt} = \begin{cases} -2 \text{ mV} & 0 < t < 2 \text{ ms} \\ 2 \text{ mV} & 2 < t < 4 \text{ ms} \end{cases}$$

Thus, the output is as sketched in Fig. 6.39.

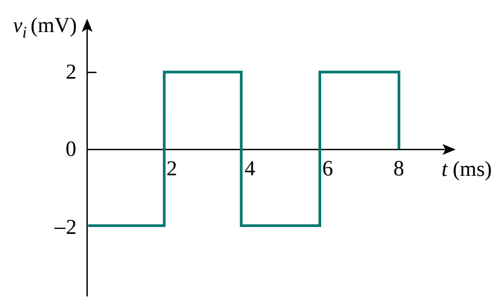


Figure 6.39 Output of the circuit in Fig. 6.38(a).

PRACTICE PROBLEM 6.14

The differentiator in Fig. 6.37 has $R = 10$ kΩ and $C = 2\ \mu$F. Given that $v_i = 3t$ V, determine the output v_o.

Answer: -60 mV.

6.6.3 Analog Computer

Op amps were initially developed for electronic analog computers. Analog computers can be programmed to solve mathematical models of mechanical or electrical systems. These models are usually expressed in terms of differential equations.

To solve simple differential equations using the analog computer requires cascading three types of op amp circuits: integrator circuits, summing amplifiers, and inverting/noninverting amplifiers for negative/

positive scaling. The best way to illustrate how an analog computer solves a differential equation is with an example.

Suppose we desire the solution $x(t)$ of the equation

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = f(t), \qquad t > 0 \tag{6.38}$$

where a, b, and c are constants, and $f(t)$ is an arbitrary forcing function. The solution is obtained by first solving the highest-order derivative term. Solving for d^2x/dt^2 yields

$$\frac{d^2x}{dt^2} = \frac{f(t)}{a} - \frac{b}{a}\frac{dx}{dt} - \frac{c}{a}x \tag{6.39}$$

To obtain dx/dt, the d^2x/dt^2 term is integrated and inverted. Finally, to obtain x, the dx/dt term is integrated and inverted. The forcing function is injected at the proper point. Thus, the analog computer for solving Eq. (6.38) is implemented by connecting the necessary summers, inverters, and integrators. A plotter or oscilloscope may be used to view the output x, or dx/dt, or d^2x/dt^2, depending on where it is connected in the system.

Although the above example is on a second-order differential equation, any differential equation can be simulated by an analog computer comprising integrators, inverters, and inverting summers. But care must be exercised in selecting the values of the resistors and capacitors, to ensure that the op amps do not saturate during the solution time interval.

The analog computers with vacuum tubes were built in the 1950s and 1960s. Recently their use has declined. They have been superseded by modern digital computers. However, we still study analog computers for two reasons. First, the availability of integrated op amps has made it possible to build analog computers easily and cheaply. Second, understanding analog computers helps with the appreciation of the digital computers.

EXAMPLE 6.15

Design an analog computer circuit to solve the differential equation:

$$\frac{d^2v_o}{dt^2} + 2\frac{dv_o}{dt} + v_o = 10\sin 4t \qquad t > 0$$

subject to $v_o(0) = -4$, $v_o'(0) = 1$, where the prime refers to the time derivative.

Solution:

We first solve for the second derivative as

$$\frac{d^2v_o}{dt^2} = 10\sin 4t - 2\frac{dv_o}{dt} - v_o \tag{6.15.1}$$

Solving this requires some mathematical operations, including summing, scaling, and integration. Integrating both sides of Eq. (6.15.1) gives

$$\frac{dv_o}{dt} = -\int_0^t \left(-10\sin 4t + 2\frac{dv_o}{dt} + v_o\right) + v_o'(0) \tag{6.15.2}$$

where $v_o'(0) = 1$. We implement Eq. (6.15.2) using the summing integrator shown in Fig. 6.40(a). The values of the resistors and capacitors have been chosen so that $RC = 1$ for the term

$$-\frac{1}{RC}\int_0^t v_o \, dt$$

Other terms in the summing integrator of Eq. (6.15.2) are implemented accordingly. The initial condition $dv_o(0)/dt = 1$ is implemented by connecting a 1-V battery with a switch across the capacitor as shown in Fig. 6.40(a).

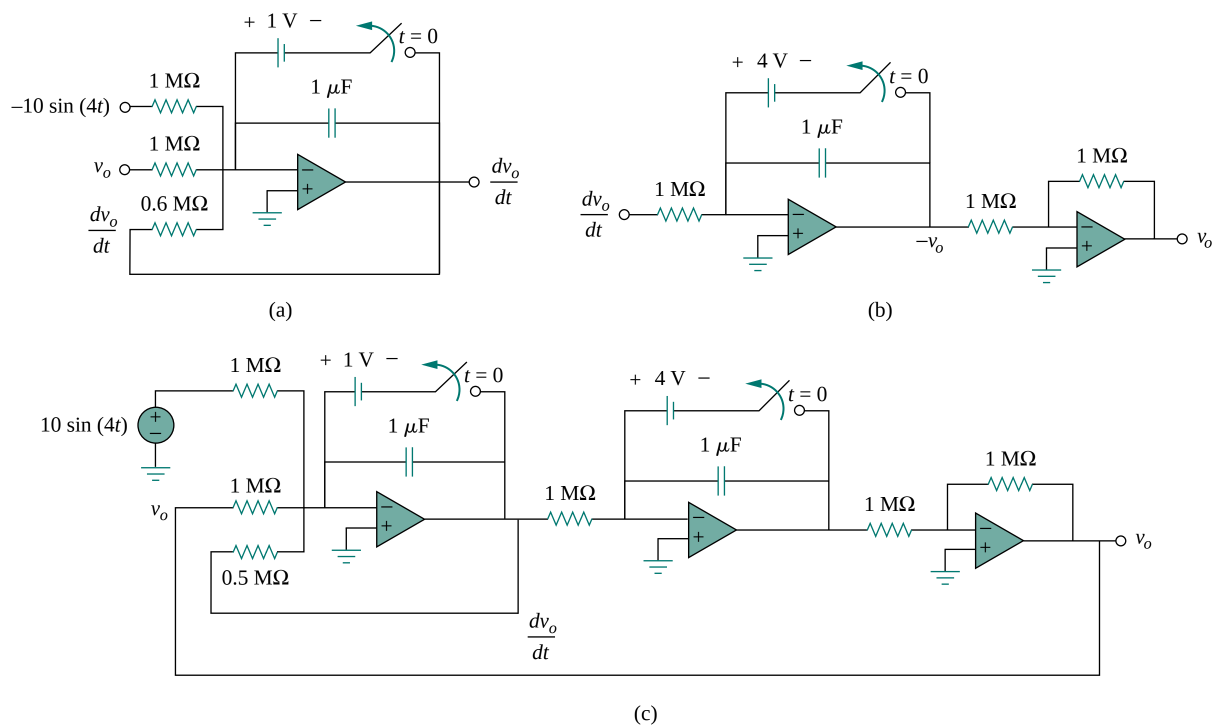


Figure 6.40 For Example 6.15.

The next step is to obtain v_o by integrating dv_o/dt and inverting the result,

$$v_o = -\int_0^t \left(-\frac{dv_o}{dt}\right) dt + v(0) \tag{6.15.3}$$

This is implemented with the circuit in Fig. 6.40(b) with the battery giving the initial condition of -4 V. We now combine the two circuits in Fig. 6.40 (a) and (b) to obtain the complete circuit shown in Fig. 6.40(c). When the input signal 10 sin $4t$ is applied, we open the switches at $t = 0$ to obtain the output waveform v_o, which may be viewed on an oscilloscope.

PRACTICE PROBLEM 6.15

Design an analog computer circuit to solve the differential equation:

$$\frac{d^2v_o}{dt^2} + 3\frac{dv_o}{dt} + 2v_o = 4\cos 10t \qquad t > 0$$

subject to $v_o(0) = 2$, $v_o'(0) = 0$.

Answer: See Fig. 6.41, where $RC = 1$ s.

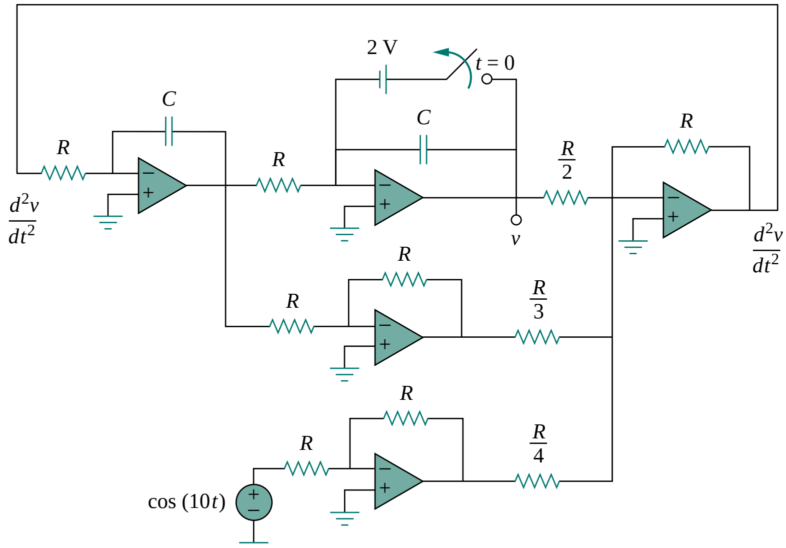


Figure 6.41 For Practice Prob. 6.15.

6.7 SUMMARY

1. The current through a capacitor is directly proportional to the time rate of change of the voltage across it.

$$i = C\frac{dv}{dt}$$

   The current through a capacitor is zero unless the voltage is changing. Thus, a capacitor acts like an open circuit to a dc source.

2. The voltage across a capacitor is directly proportional to the time integral of the current through it.

$$v = \frac{1}{C}\int_{-\infty}^{t} i\,dt = \frac{1}{C}\int_{t_0}^{t} i\,dt + i(t_0)$$

   The voltage across a capacitor cannot change instantly.

3. Capacitors in series and in parallel are combined in the same way as conductances.

4. The voltage across an inductor is directly proportional to the time rate of change of the current through it.

$$v = L\frac{di}{dt}$$

The voltage across the inductor is zero unless the current is changing. Thus an inductor acts like a short circuit to a dc source.

5. The current through an inductor is directly proportional to the time integral of the voltage across it.

$$i = \frac{1}{L}\int_{-\infty}^{t} v\,dt = \frac{1}{L}\int_{t_0}^{t} v\,dt + v(t_0)$$

The current through an inductor cannot change instantly.

6. Inductors in series and in parallel are combined in the same way resistors in series and in parallel are combined.
7. At any given time t, the energy stored in a capacitor is $\frac{1}{2}Cv^2$, while the energy stored in an inductor is $\frac{1}{2}Li^2$.
8. Three application circuits, the integrator, the differentiator, and the analog computer, can be realized using resistors, capacitors, and op amps.

REVIEW QUESTIONS

6.1 What charge is on a 5-F capacitor when it is connected across a 120-V source?
(a) 600 C (b) 300 C
(c) 24 C (d) 12 C

6.2 Capacitance is measured in:
(a) coulombs (b) joules
(c) henrys (d) farads

6.3 When the total charge in a capacitor is doubled, the energy stored:
(a) remains the same (b) is halved
(c) is doubled (d) is quadrupled

6.4 Can the voltage waveform in Fig. 6.42 be associated with a capacitor?
(a) Yes (b) No

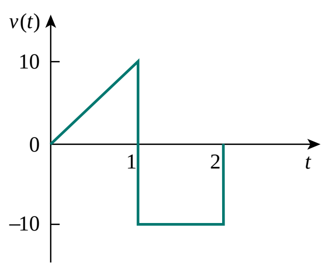


Figure 6.42 For Review Question 6.4.

6.5 The total capacitance of two 40-mF series-connected capacitors in parallel with a 4-mF capacitor is:
(a) 3.8 mF (b) 5 mF (c) 24 mF
(d) 44 mF (e) 84 mF

6.6 In Fig. 6.43, if $i = \cos 4t$ and $v = \sin 4t$, the element is:
(a) a resistor (b) a capacitor (c) an inductor

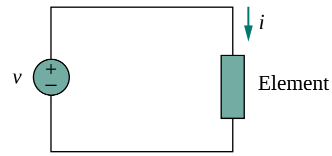


Figure 6.43 For Review Question 6.6.

6.7 A 5-H inductor changes its current by 3 A in 0.2 s. The voltage produced at the terminals of the inductor is:
(a) 75 V (b) 8.888 V
(c) 3 V (d) 1.2 V

6.8 If the current through a 10-mH inductor increases from zero to 2 A, how much energy is stored in the inductor?
(a) 40 mJ (b) 20 mJ
(c) 10 mJ (d) 5 mJ

6.9 Inductors in parallel can be combined just like resistors in parallel.

(a) True (b) False

6.10 For the circuit in Fig. 6.44, the voltage divider formula is:

(a) $v_1 = \dfrac{L_1 + L_2}{L_1} v_s$ (b) $v_1 = \dfrac{L_1 + L_2}{L_2} v_s$

(c) $v_1 = \dfrac{L_2}{L_1 + L_2} v_s$ (d) $v_1 = \dfrac{L_1}{L_1 + L_2} v_s$

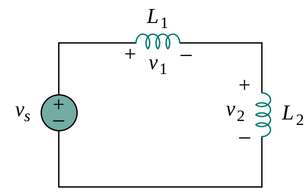


Figure 6.44 For Review Question 6.10.

Answers: 6.1a, 6.2d, 6.3d, 6.4b, 6.5c, 6.6b, 6.7a, 6.8b, 6.9a, 6.10d.

PROBLEMS

Section 6.2 Capacitors

6.1 If the voltage across a 5-F capacitor is $2te^{-3t}$ V, find the current and the power.

6.2 A 40-μF capacitor is charged to 120 V and is then allowed to discharge to 80 V. How much energy is lost?

6.3 In 5 s, the voltage across a 40-mF capacitor changes from 160 V to 220 V. Calculate the average current through the capacitor.

6.4 A current of $6 \sin 4t$ A flows through a 2-F capacitor. Find the voltage $v(t)$ across the capacitor given that $v(0) = 1$ V.

6.5 If the current waveform in Fig. 6.45 is applied to a 20-μF capacitor, find the voltage $v(t)$ across the capacitor. Assume that $v(0) = 0$.

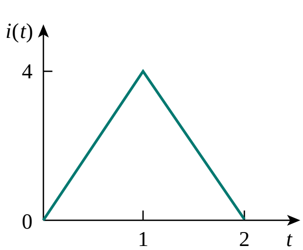


Figure 6.45 For Prob. 6.5.

6.6 The voltage waveform in Fig. 6.46 is applied across a 30-μF capacitor. Draw the current waveform through it.

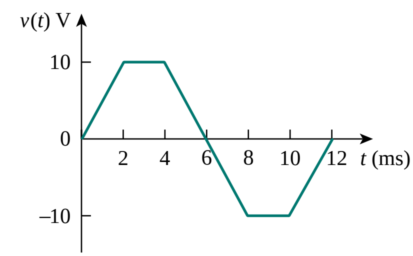


Figure 6.46 For Prob. 6.6.

6.7 At $t = 0$, the voltage across a 50-mF capacitor is 10 V. Calculate the voltage across the capacitor for $t > 0$ when current $4t$ mA flows through it.

6.8 The current through a 0.5-F capacitor is $6(1 - e^{-t})$ A. Determine the voltage and power at $t = 2$ s. Assume $v(0) = 0$.

6.9 If the voltage across a 2-F capacitor is as shown in Fig. 6.47, find the current through the capacitor.

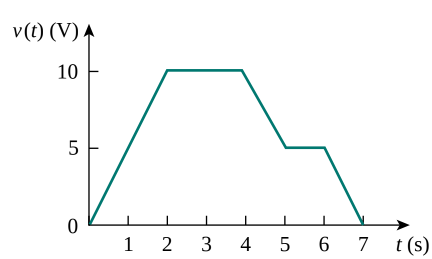


Figure 6.47 For Prob. 6.9.

6.10 The current through an initially uncharged 4-μF capacitor is shown in Fig. 6.48. Find the voltage across the capacitor for $0 < t < 3$.

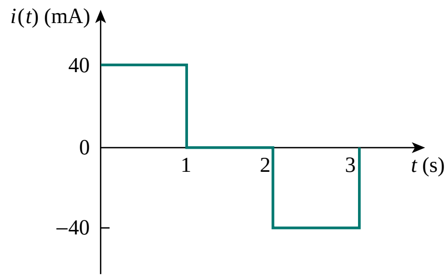


Figure 6.48 For Prob. 6.10.

6.11 A voltage of $60 \cos 4\pi t$ V appears across the terminals of a 3-mF capacitor. Calculate the current through the capacitor and the energy stored in it from $t = 0$ to $t = 0.125$ s.

6.12 Find the voltage across the capacitors in the circuit of Fig. 6.49 under dc conditions.

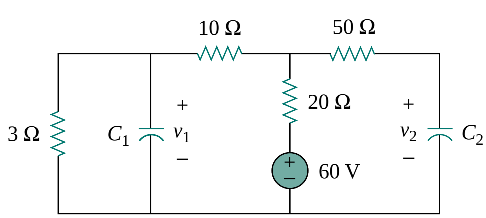


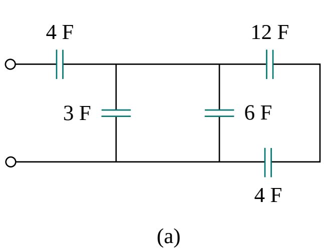
Figure 6.49 For Prob. 6.12.

Section 6.3 Series and Parallel Capacitors

6.13 What is the total capacitance of four 30-mF capacitors connected in:
(a) parallel (b) series

6.14 Two capacitors (20 μF and 30 μF) are connected to a 100-V source. Find the energy stored in each capacitor if they are connected in:
(a) parallel (b) series

6.15 Determine the equivalent capacitance for each of the circuits in Fig. 6.50.



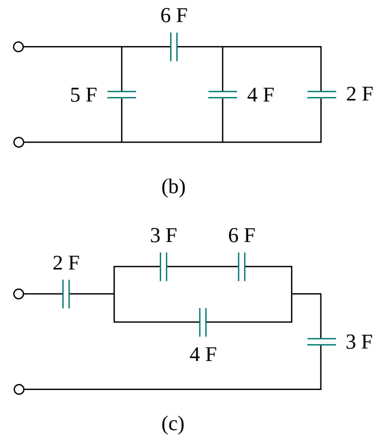


Figure 6.50 For Prob. 6.15.

6.16 Find C_{eq} for the circuit in Fig. 6.51.

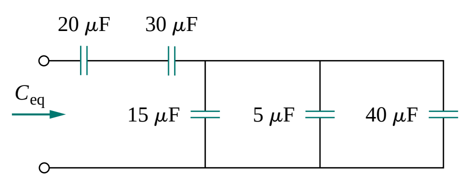


Figure 6.51 For Prob. 6.16.

6.17 Calculate the equivalent capacitance for the circuit in Fig. 6.52. All capacitances are in mF.

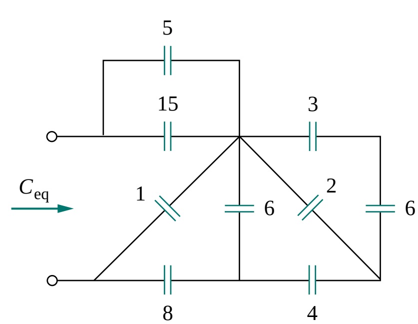


Figure 6.52 For Prob. 6.17.

6.18 Determine the equivalent capacitance at terminals a-b of the circuit in Fig. 6.53.

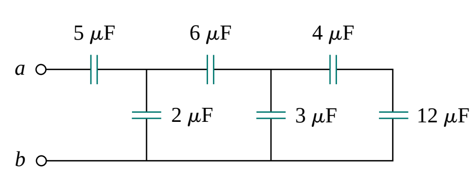


Figure 6.53 For Prob. 6.18.

6.19 Obtain the equivalent capacitance of the circuit in Fig. 6.54.

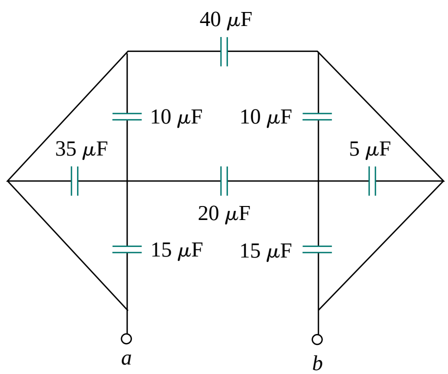


Figure 6.54 For Prob. 6.19.

6.20 For the circuit in Fig. 6.55, determine:

(a) the voltage across each capacitor,

(b) the energy stored in each capacitor.

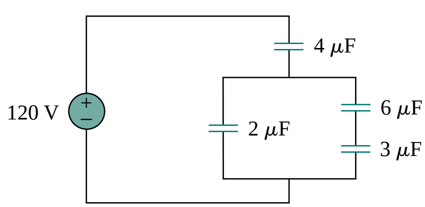


Figure 6.55 For Prob. 6.20.

6.21 Repeat Prob. 6.20 for the circuit in Fig. 6.56.

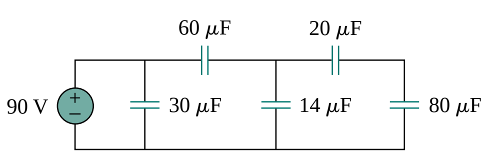


Figure 6.56 For Prob. 6.21.

6.22 (a) Show that the voltage-division rule for two capacitors in series as in Fig. 6.57(a) is

$$v_1 = \frac{C_2}{C_1 + C_2} v_s, \qquad v_2 = \frac{C_1}{C_1 + C_2} v_s$$

assuming that the initial conditions are zero.

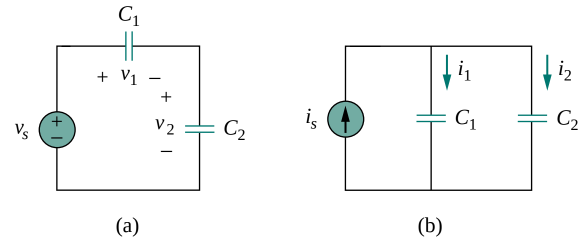


Figure 6.57 For Prob. 6.22.

(b) For two capacitors in parallel as in Fig. 6.57(b), show that the current-division rule is

$$i_1 = \frac{C_1}{C_1 + C_2} i_s, \qquad i_2 = \frac{C_2}{C_1 + C_2} i_s$$

assuming that the initial conditions are zero.

6.23 Three capacitors, $C_1 = 5\ \mu$F, $C_2 = 10\ \mu$F, and $C_3 = 20\ \mu$F, are connected in parallel across a 150-V source. Determine:

(a) the total capacitance,

(b) the charge on each capacitor,

(c) the total energy stored in the parallel combination.

6.24 The three capacitors in the previous problem are placed in series with a 200-V source. Compute:

(a) the total capacitance,

(b) the charge on each capacitor,

(c) the total energy stored in the series combination.

***6.25** Obtain the equivalent capacitance of the network shown in Fig. 6.58.

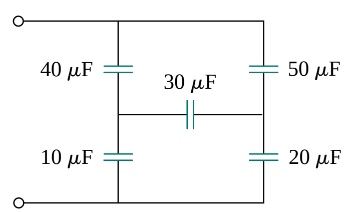


Figure 6.58 For Prob. 6.25.

6.26 Determine C_{eq} for each circuit in Fig. 6.59.

*An asterisk indicates a challenging problem.

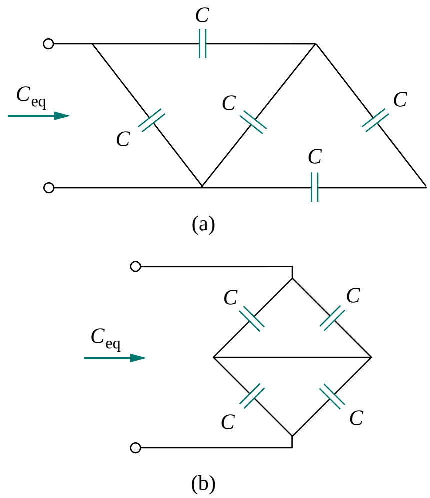


Figure 6.59 For Prob. 6.26.

6.27 Assuming that the capacitors are initially uncharged, find $v_o(t)$ in the circuit in Fig. 6.60.

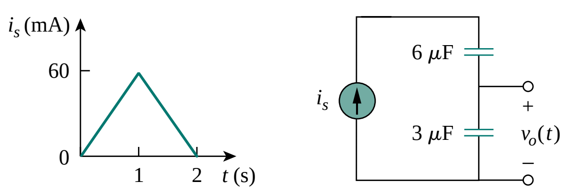


Figure 6.60 For Prob. 6.27.

6.28 If $v(0) = 0$, find $v(t)$, $i_1(t)$, and $i_2(t)$ in the circuit in Fig. 6.61.

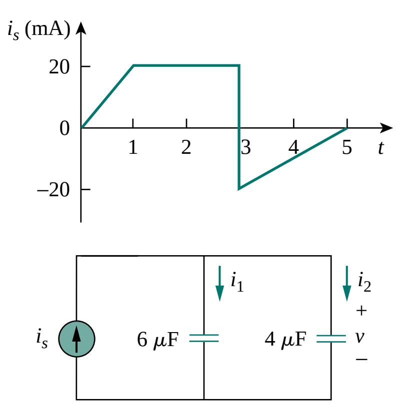


Figure 6.61 For Prob. 6.28.

6.29 For the circuit in Fig. 6.62, let $v = 10e^{-3t}$ V and $v_1(0) = 2$ V. Find:

(a) $v_2(0)$ (b) $v_1(t)$ and $v_2(t)$
(c) $i(t)$, $i_1(t)$, and $i_2(t)$

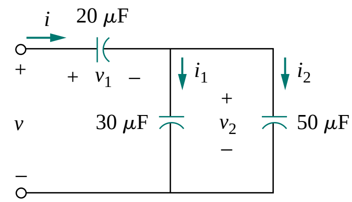


Figure 6.62 For Prob. 6.29.

Section 6.4 Inductors

6.30 The current through a 10-mH inductor is $6e^{-t/2}$ A. Find the voltage and the power at $t = 3$ s.

6.31 The current in a coil increases uniformly from 0.4 to 1 A in 2 s so that the voltage across the coil is 60 mV. Calculate the inductance of the coil.

6.32 The current through a 0.25-mH inductor is $12 \cos 2t$ A. Determine the terminal voltage and the power.

6.33 The current through a 12-mH inductor is $4 \sin 100t$ A. Find the voltage, and also the energy stored in the inductor for $0 < t < \pi/200$ s.

6.34 The current through a 40-mH inductor is

$$i(t) = \begin{cases} 0, & t < 0 \\ te^{-2t} \text{ A}, & t > 0 \end{cases}$$

Find the voltage $v(t)$.

6.35 The voltage across a 2-H inductor is $20(1 - e^{-2t})$ V. If the initial current through the inductor is 0.3 A, find the current and the energy stored in the inductor at $t = 1$ s.

6.36 If the voltage waveform in Fig. 6.63 is applied across the terminals of a 5-H inductor, calculate the current through the inductor. Assume $i(0) = -1$ A.

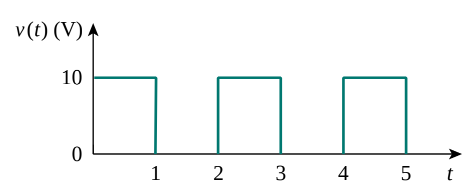


Figure 6.63 For Prob. 6.36.

6.37 The current in an 80-mH inductor increases from 0 to 60 mA. How much energy is stored in the inductor?

6.38 A voltage of $(4 + 10 \cos 2t)$ V is applied to a 5-H inductor. Find the current $i(t)$ through the inductor if $i(0) = -1$ A.

6.39 If the voltage waveform in Fig. 6.64 is applied to a 10-mH inductor, find the inductor current $i(t)$. Assume $i(0) = 0$.

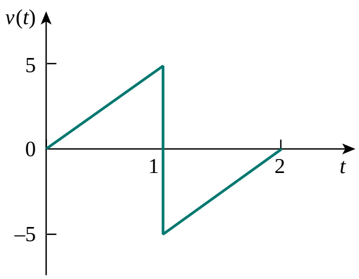


Figure 6.64 For Prob. 6.39.

6.40 Find v_C, i_L, and the energy stored in the capacitor and inductor in the circuit of Fig. 6.65 under dc conditions.

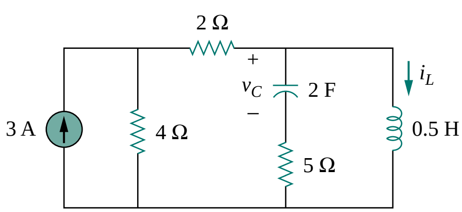


Figure 6.65 For Prob. 6.40.

6.41 For the circuit in Fig. 6.66, calculate the value of R that will make the energy stored in the capacitor the same as that stored in the inductor under dc conditions.

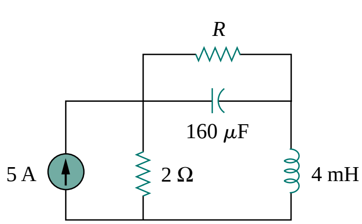


Figure 6.66 For Prob. 6.41.

6.42 Under dc conditions, find the voltage across the capacitors and the current through the inductors in the circuit of Fig. 6.67.

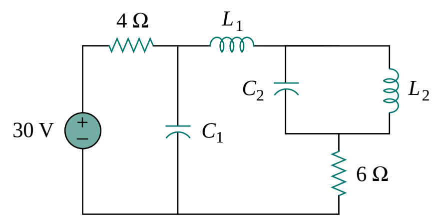


Figure 6.67 For Prob. 6.42.

Section 6.5 Series and Parallel Inductors

6.43 Find the equivalent inductance for each circuit in Fig. 6.68.

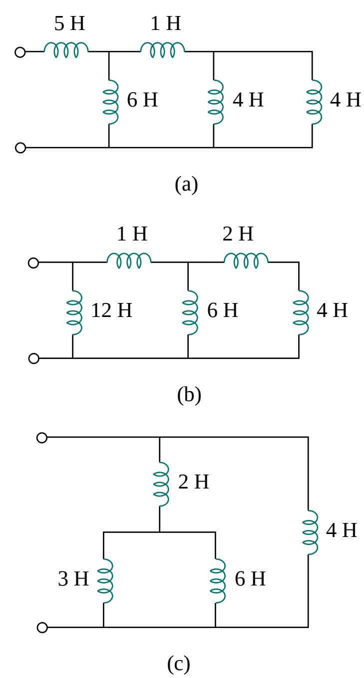


Figure 6.68 For Prob. 6.43.

6.44 Obtain L_{eq} for the inductive circuit of Fig. 6.69. All inductances are in mH.

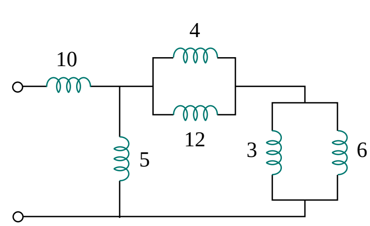


Figure 6.69 For Prob. 6.44.

6.45 Determine L_{eq} at terminals *a*-*b* of the circuit in Fig. 6.70.

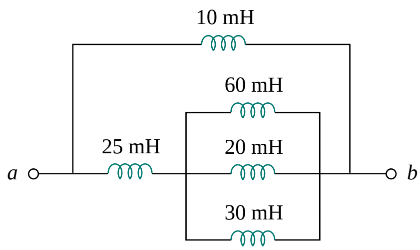


Figure 6.70 For Prob. 6.45.

6.46 Find L_{eq} at the terminals of the circuit in Fig. 6.71.

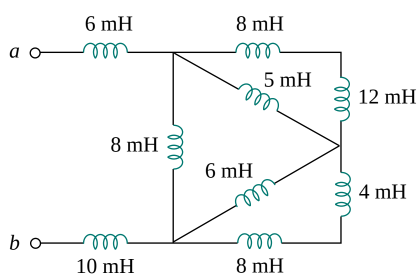


Figure 6.71 For Prob. 6.46.

6.47 Find the equivalent inductance looking into the terminals of the circuit in Fig. 6.72.

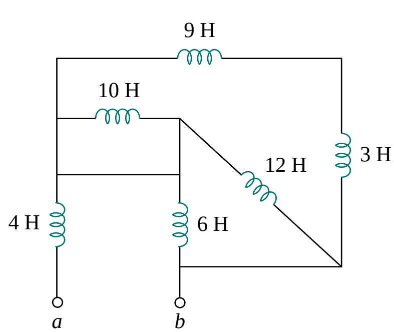


Figure 6.72 For Prob. 6.47.

6.48 Determine L_{eq} in the circuit in Fig. 6.73.

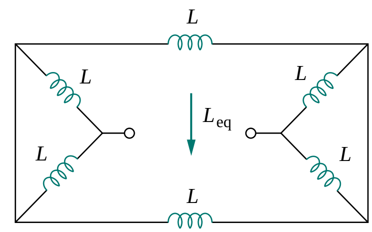


Figure 6.73 For Prob. 6.48.

6.49 Find L_{eq} in the circuit in Fig. 6.74.

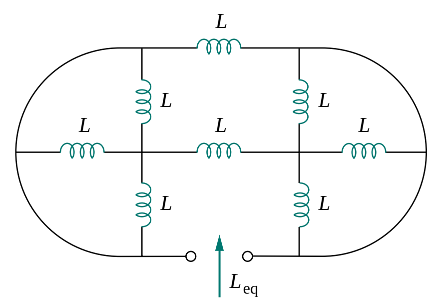


Figure 6.74 For Prob. 6.49.

**6.50* Determine L_{eq} that may be used to represent the inductive network of Fig. 6.75 at the terminals.

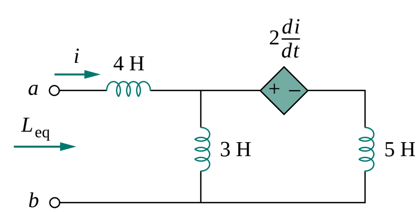


Figure 6.75 For Prob. 6.50.

6.51 The current waveform in Fig. 6.76 flows through a 3-H inductor. Sketch the voltage across the inductor over the interval $0 < t < 6$ s.

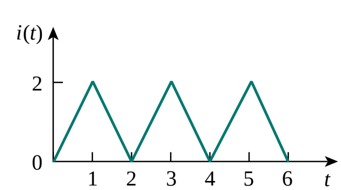


Figure 6.76 For Prob. 6.51.

6.52 (a) For two inductors in series as in Fig. 6.77(b), show that the current-division principle is

$$v_1 = \frac{L_1}{L_1 + L_2} v_s, \qquad v_2 = \frac{L_2}{L_1 + L_2} v_s$$

assuming that the initial conditions are zero.

(b) For two inductors in parallel as in Fig. 6.77(b), show that the current-division principle is

$$i_1 = \frac{L_2}{L_1 + L_2} i_s, \qquad i_2 = \frac{L_1}{L_1 + L_2} i_s$$

assuming that the initial conditions are zero.

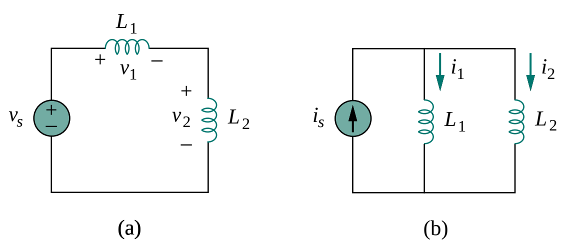


Figure 6.77 For Prob. 6.52.

6.53 In the circuit of Fig. 6.78, let $i_s(t) = 6e^{-2t}$ mA, $t \geq 0$ and $i_1(0) = 4$ mA. Find:

(a) $i_2(0)$,

(b) $i_1(t)$ and $i_2(t)$, $t > 0$,

(c) $v_1(t)$ and $v_2(t)$, $t > 0$,

(d) the energy in each inductor at $t = 0.5$ s.

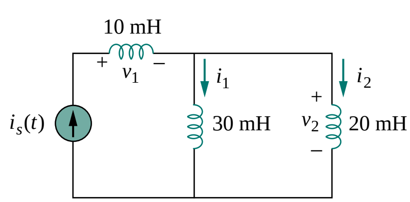


Figure 6.78 For Prob. 6.53.

6.54 The inductors in Fig. 6.79 are initially charged and are connected to the black box at $t = 0$. If $i_1(0) = 4$ A, $i_2(0) = -2$ A, and $v(t) = 50e^{-200t}$ mV, $t \geq 0$, find:

(a) the energy initially stored in each inductor,

(b) the total energy delivered to the black box from $t = 0$ to $t = \infty$,

(c) $i_1(t)$ and $i_2(t)$, $t \geq 0$,

(d) $i(t)$, $t \geq 0$.

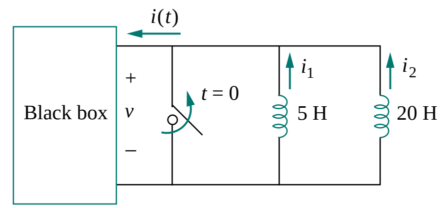


Figure 6.79 For Prob. 6.54.

6.55 Find i and v in the circuit of Fig. 6.80 assuming that $i(0) = 0 = v(0)$.

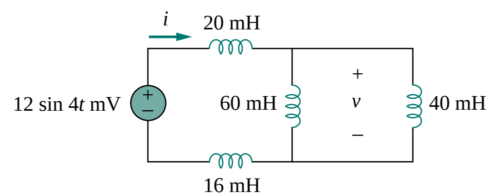


Figure 6.80 For Prob. 6.55.

Section 6.6 Applications

6.56 An op amp integrator has $R = 50$ kΩ and $C = 0.04$ μF. If the input voltage is $v_i = 10 \sin 50t$ mV, obtain the output voltage.

6.57 A 10-V dc voltage is applied to an integrator with $R = 50$ kΩ, $C = 100$ μF at $t = 0$. How long will it take for the op amp to saturate if the saturation voltages are $+12$ V and -12 V? Assume that the initial capacitor voltage was zero.

6.58 An op amp integrator with $R = 4$ MΩ and $C = 1$ μF has the input waveform shown in Fig. 6.81. Plot the output waveform.

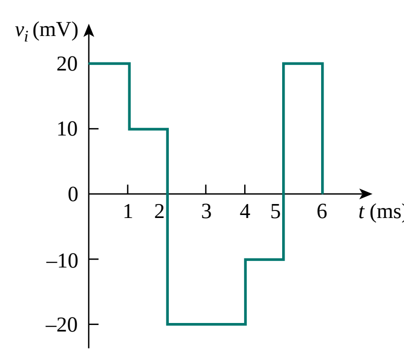


Figure 6.81 For Prob. 6.58.

6.59 Using a single op amp, a capacitor, and resistors of 100 kΩ or less, design a circuit to implement

$$v_o = -50 \int_0^t v_i(t)\, dt$$

Assume $v_o = 0$ at $t = 0$.

6.60 Show how you would use a single op amp to generate

$$v_o = -\int_0^t (v_1 + 4v_2 + 10v_3)\,dt$$

If the integrating capacitor is $C = 2\ \mu$F, obtain other component values.

6.61 At $t = 1.5$ ms, calculate v_o due to the cascaded integrators in Fig. 6.82. Assume that the integrators are reset to 0 V at $t = 0$.

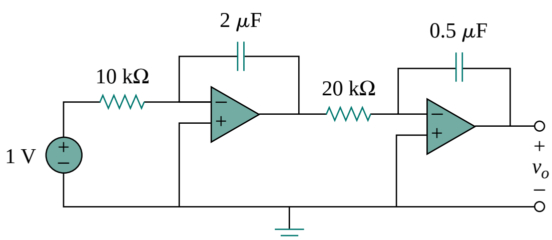


Figure 6.82 For Prob. 6.61.

6.62 Show that the circuit in Fig. 6.83 is a noninverting integrator.

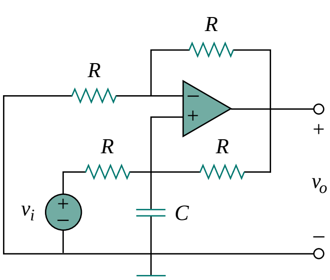
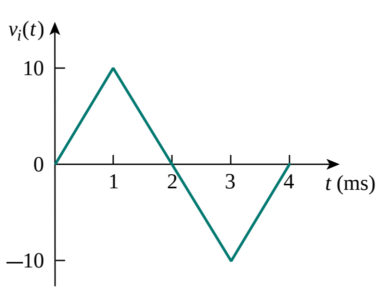
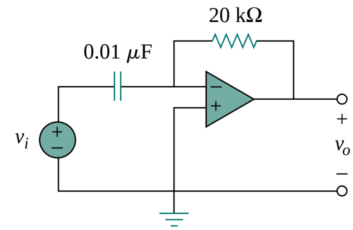


Figure 6.83 For Prob. 6.62.

6.63 The triangular waveform in Fig. 6.84(a) is applied to the input of the op amp differentiator in Fig. 6.84(b). Plot the output.



(a)



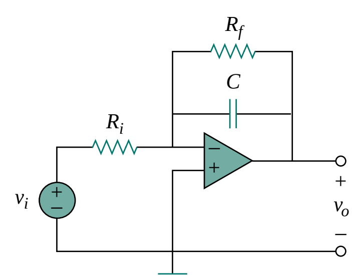
(b)

Figure 6.84 For Prob. 6.63.

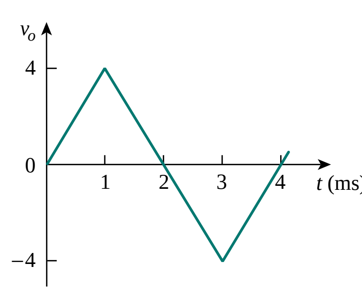
6.64 An op amp differentiator has $R = 250\ k\Omega$ and $C = 10\ \mu$F. The input voltage is a ramp $r(t) = 12t$ mV. Find the output voltage.

6.65 A voltage waveform has the following characteristics: a positive slope of 20 V/s for 5 ms followed by a negative slope of 10 V/s for 10 ms. If the waveform is applied to a differentiator with $R = 50$ kΩ, $C = 10\ \mu$F, sketch the output voltage waveform.

***6.66** The output v_o of the op amp circuit of Fig. 6.85(a) is shown in Fig. 6.85(b). Let $R_i = R_f = 1$ MΩ and $C = 1\ \mu$F. Determine the input voltage waveform and sketch it.



(a)



(b)

Figure 6.85 For Prob. 6.66.

6.67 Design an analog computer to simulate

$$\frac{d^2 v_o}{dt^2} + 2\frac{dv_o}{dt} + v_o = 10 \sin 2t$$

where $v_0(0) = 2$ and $v_0'(0) = 0$.

6.68 Design an analog computer to solve the differential equation

$$\frac{di(t)}{dt} + 3i(t) = 2 \qquad t > 0$$

and assume that $i(0) = 1$ mA.

6.69 Figure 6.86 presents an analog computer designed to solve a differential equation. Assuming $f(t)$ is known, set up the equation for $f(t)$.

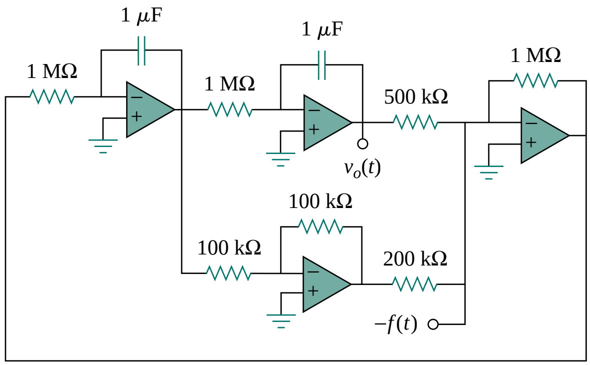


Figure 6.86 For Prob. 6.69.

COMPREHENSIVE PROBLEMS

6.70 Your laboratory has available a large number of 10-μF capacitors rated at 300 V. To design a capacitor bank of 40-μF rated at 600 V, how many 10-μF capacitors are needed and how would you connect them?

6.71 When a capacitor is connected to a dc source, its voltage rises from 20 V to 36 V in 4 μs with an average charging current of 0.6 A. Determine the value of the capacitance.

6.72 A square-wave generator produces the voltage waveform shown in Fig. 6.87(a). What kind of a circuit component is needed to convert the voltage waveform to the triangular current waveform shown in Fig. 6.87(b)? Calculate the value of the component, assuming that it is initially uncharged.

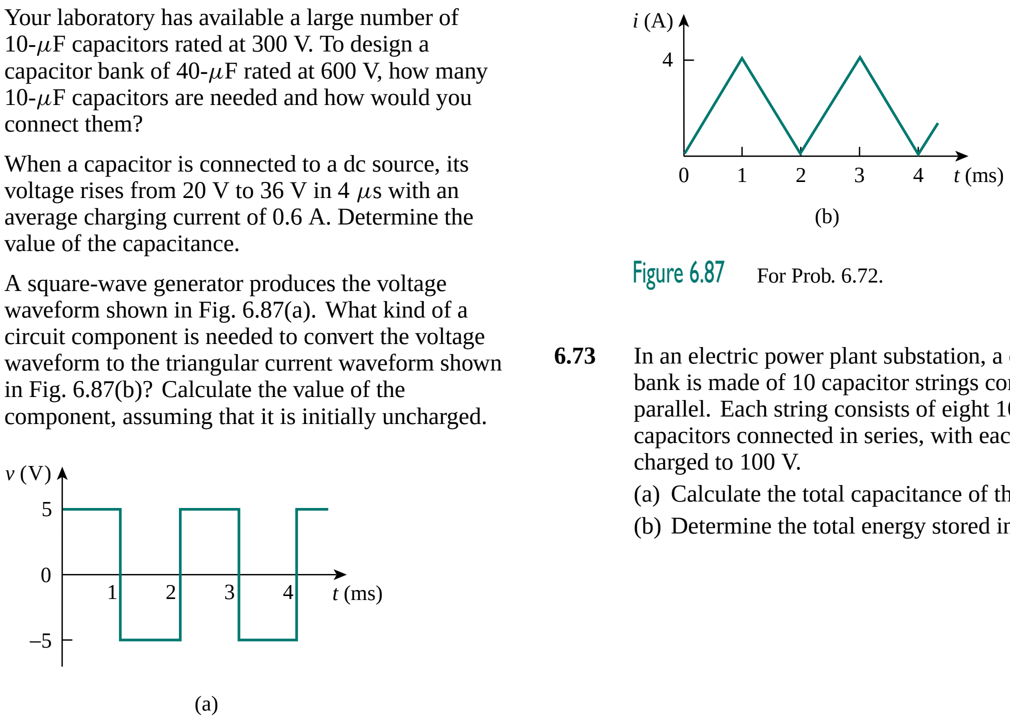


Figure 6.87 For Prob. 6.72.

6.73 In an electric power plant substation, a capacitor bank is made of 10 capacitor strings connected in parallel. Each string consists of eight 1000-μF capacitors connected in series, with each capacitor charged to 100 V.

(a) Calculate the total capacitance of the bank.

(b) Determine the total energy stored in the bank.