PART 2

AC CIRCUITS

Chapter 9 *Sinusoids and Phasors*

Chapter 10 *Sinusoidal Steady-State Analysis*

Chapter 11 *AC Power Analysis*

Chapter 12 *Three-Phase Circuits*

Chapter 13 *Magnetically Coupled Circuits*

Chapter 14 *Frequency Response*

CHAPTER 9

SINUSOIDS AND PHASORS

The desire to understand the world and the desire to reform it are the two great engines of progress.

— Bertrand Russell

Historical Profiles

Heinrich Rudorf Hertz (1857–1894), a German experimental physicist, demonstrated that electromagnetic waves obey the same fundamental laws as light. His work confirmed James Clerk Maxwell's celebrated 1864 theory and prediction that such waves existed.

Hertz was born into a prosperous family in Hamburg, Germany. He attended the University of Berlin and did his doctorate under the prominent physicist Hermann von Helmholtz. He became a professor at Karlsruhe, where he began his quest for electromagnetic waves. Hertz successfully generated and detected electromagnetic waves; he was the first to show that light is electromagnetic energy. In 1887, Hertz noted for the first time the photoelectric effect of electrons in a molecular structure. Although Hertz only lived to the age of 37, his discovery of electromagnetic waves paved the way for the practical use of such waves in radio, television, and other communication systems. The unit of frequency, the hertz, bears his name.

Charles Proteus Steinmetz (1865–1923), a German-Austrian mathematician and engineer, introduced the phasor method (covered in this chapter) in ac circuit analysis. He is also noted for his work on the theory of hysteresis.

Steinmetz was born in Breslau, Germany, and lost his mother at the age of one. As a youth, he was forced to leave Germany because of his political activities just as he was about to complete his doctoral dissertation in mathematics at the University of Breslau. He migrated to Switzerland and later to the United States, where he was employed by General Electric in 1893. That same year, he published a paper in which complex numbers were used to analyze ac circuits for the first time. This led to one of his many textbooks, *Theory and Calculation of ac Phenomena*, published by McGraw-Hill in 1897. In 1901, he became the president of the American Institute of Electrical Engineers, which later became the IEEE.

9.1 INTRODUCTION

Thus far our analysis has been limited for the most part to dc circuits: those circuits excited by constant or time-invariant sources. We have restricted the forcing function to dc sources for the sake of simplicity, for pedagogic reasons, and also for historic reasons. Historically, dc sources were the main means of providing electric power up until the late 1800s. At the end of that century, the battle of direct current versus alternating current began. Both had their advocates among the electrical engineers of the time. Because ac is more efficient and economical to transmit over long distances, ac systems ended up the winner. Thus, it is in keeping with the historical sequence of events that we considered dc sources first.

We now begin the analysis of circuits in which the source voltage or current is time-varying. In this chapter, we are particularly interested in sinusoidally time-varying excitation, or simply, excitation by a *sinusoid*.

> A sinusoid is a signal that has the form of the sine or cosine function.

A sinusoidal current is usually referred to as *alternating current* (*ac*). Such a current reverses at regular time intervals and has alternately positive and negative values. Circuits driven by sinusoidal current or voltage sources are called *ac circuits*.

We are interested in sinusoids for a number of reasons. First, nature itself is characteristically sinusoidal. We experience sinusoidal variation in the motion of a pendulum, the vibration of a string, the ripples on the ocean surface, the political events of a nation, the economic fluctuations of the stock market, and the natural response of underdamped second-order systems, to mention but a few. Second, a sinusoidal signal is easy to generate and transmit. It is the form of voltage generated throughout the world and supplied to homes, factories, laboratories, and so on. It is the dominant form of signal in the communications and electric power industries. Third, through Fourier analysis, any practical periodic signal can be represented by a sum of sinusoids. Sinusoids, therefore, play an important role in the analysis of periodic signals. Lastly, a sinusoid is easy to handle mathematically. The derivative and integral of a sinusoid are themselves sinusoids. For these and other reasons, the sinusoid is an extremely important function in circuit analysis.

A sinusoidal forcing function produces both a natural (or transient) response and a forced (or steady-state) response, much like the step function, which we studied in Chapters 7 and 8. The natural response of a circuit is dictated by the nature of the circuit, while the steady-state response always has a form similar to the forcing function. However, the natural response dies out with time so that only the steady-state response remains after a long time. When the natural response has become negligibly small compared with the steady-state response, we say that the circuit is operating at sinusoidal steady state. It is this *sinusoidal steady-state response* that is of main interest to us in this chapter.

We begin with a basic discussion of sinusoids and phasors. We then introduce the concepts of impedance and admittance. The basic circuit laws, Kirchhoff's and Ohm's, introduced for dc circuits, will be applied to ac circuits. Finally, we consider applications of ac circuits in phase-shifters and bridges.

9.2 SINUSOIDS

Consider the sinusoidal voltage

$$v(t) = V_m \sin \omega t \tag{9.1}$$

where

V_m = the *amplitude* of the sinusoid
ω = the *angular frequency* in radians/s
ωt = the *argument* of the sinusoid

The sinusoid is shown in Fig. 9.1(a) as a function of its argument and in Fig. 9.1(b) as a function of time. It is evident that the sinusoid repeats itself every T seconds; thus, T is called the *period* of the sinusoid. From the two plots in Fig. 9.1, we observe that $\omega T = 2\pi$,

$$\boxed{T = \frac{2\pi}{\omega}} \tag{9.2}$$

The fact that $v(t)$ repeats itself every T seconds is shown by replacing t by $t + T$ in Eq. (9.1). We get

$$\begin{aligned} v(t + T) &= V_m \sin \omega (t + T) = V_m \sin \omega \left(t + \frac{2\pi}{\omega}\right) \\ &= V_m \sin(\omega t + 2\pi) = V_m \sin \omega t = v(t) \end{aligned} \tag{9.3}$$

Hence,

$$\boxed{v(t + T) = v(t)} \tag{9.4}$$

that is, v has the same value at $t + T$ as it does at t and $v(t)$ is said to be *periodic*. In general,

A periodic function is one that satisfies $f(t) = f(t + nT)$, for all t and for all integers n.

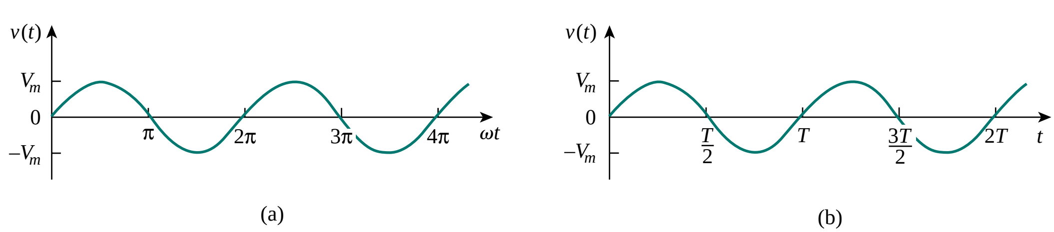

Figure 9.1 A sketch of $V_m \sin \omega t$: (a) as a function of ωt, (b) as a function of t.

As mentioned, the *period* T of the periodic function is the time of one complete cycle or the number of seconds per cycle. The reciprocal of this quantity is the number of cycles per second, known as the *cyclic frequency* f of the sinusoid. Thus,

$$\boxed{f = \frac{1}{T}} \tag{9.5}$$

From Eqs. (9.2) and (9.5), it is clear that

$$\omega = 2\pi f \tag{9.6}$$

The unit of f is named after the German physicist Heinrich R. Hertz (1857–1894).

While ω is in radians per second (rad/s), f is in hertz (Hz).

Let us now consider a more general expression for the sinusoid,

$$v(t) = V_m \sin(\omega t + \phi) \tag{9.7}$$

where $(\omega t + \phi)$ is the argument and ϕ is the *phase*. Both argument and phase can be in radians or degrees.

Let us examine the two sinusoids

$$v_1(t) = V_m \sin \omega t \qquad \text{and} \qquad v_2(t) = V_m \sin(\omega t + \phi) \tag{9.8}$$

shown in Fig. 9.2. The starting point of v_2 in Fig. 9.2 occurs first in time. Therefore, we say that v_2 *leads* v_1 by ϕ or that v_1 *lags* v_2 by ϕ. If $\phi \neq 0$, we also say that v_1 and v_2 are *out of phase*. If $\phi = 0$, then v_1 and v_2 are said to be *in phase*; they reach their minima and maxima at exactly the same time. We can compare v_1 and v_2 in this manner because they operate at the same frequency; they do not need to have the same amplitude.

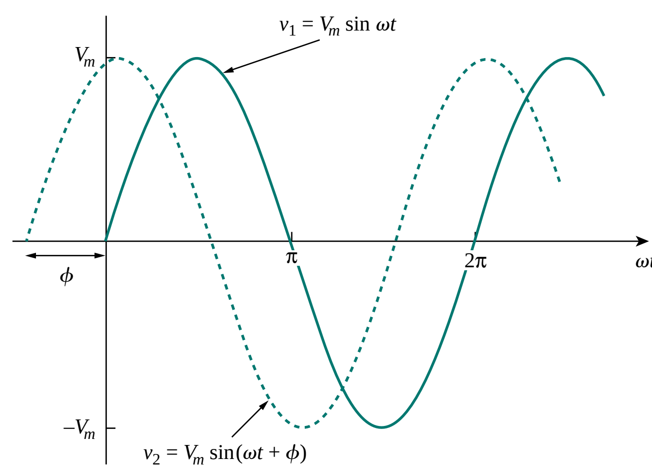

Figure 9.2 Two sinusoids with different phases.

A sinusoid can be expressed in either sine or cosine form. When comparing two sinusoids, it is expedient to express both as either sine or cosine with positive amplitudes. This is achieved by using the following trigonometric identities:

$$\begin{aligned} \sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\ \cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \end{aligned} \tag{9.9}$$

With these identities, it is easy to show that

$$\begin{aligned}\sin(\omega t \pm 180^\circ) &= -\sin \omega t \\ \cos(\omega t \pm 180^\circ) &= -\cos \omega t \\ \sin(\omega t \pm 90^\circ) &= \pm \cos \omega t \\ \cos(\omega t \pm 90^\circ) &= \mp \sin \omega t\end{aligned} \tag{9.10}$$

Using these relationships, we can transform a sinusoid from sine form to cosine form or vice versa.

A graphical approach may be used to relate or compare sinusoids as an alternative to using the trigonometric identities in Eqs. (9.9) and (9.10). Consider the set of axes shown in Fig. 9.3(a). The horizontal axis represents the magnitude of cosine, while the vertical axis (pointing down) denotes the magnitude of sine. Angles are measured positively counterclockwise from the horizontal, as usual in polar coordinates. This graphical technique can be used to relate two sinusoids. For example, we see in Fig. 9.3(a) that subtracting 90° from the argument of $\cos \omega t$ gives $\sin \omega t$, or $\cos(\omega t - 90^\circ) = \sin \omega t$. Similarly, adding 180° to the argument of $\sin \omega t$ gives $-\sin \omega t$, or $\sin(\omega t - 180^\circ) = -\sin \omega t$, as shown in Fig. 9.3(b).

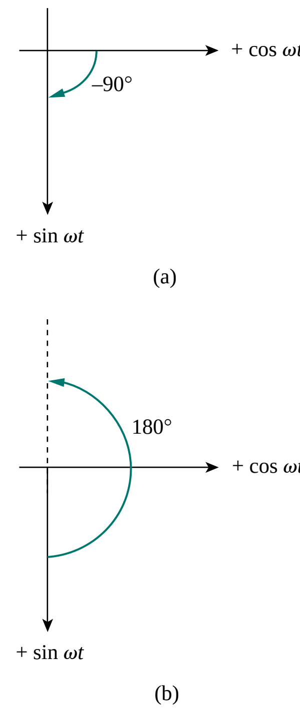

Figure 9.3 A graphical means of relating cosine and sine: (a) $\cos(\omega t - 90^\circ) = \sin \omega t$, (b) $\sin(\omega t + 180^\circ) = -\sin \omega t$.

The graphical technique can also be used to add two sinusoids of the same frequency when one is in sine form and the other is in cosine form. To add $A \cos \omega t$ and $B \sin \omega t$, we note that A is the magnitude of $\cos \omega t$ while B is the magnitude of $\sin \omega t$, as shown in Fig. 9.4(a). The magnitude and argument of the resultant sinusoid in cosine form is readily obtained from the triangle. Thus,

$$A \cos \omega t + B \sin \omega t = C \cos(\omega t - \theta) \tag{9.11}$$

where

$$C = \sqrt{A^2 + B^2}, \qquad \theta = \tan^{-1} \frac{B}{A} \tag{9.12}$$

For example, we may add $3 \cos \omega t$ and $-4 \sin \omega t$ as shown in Fig. 9.4(b) and obtain

$$3 \cos \omega t - 4 \sin \omega t = 5 \cos(\omega t + 53.1^\circ) \tag{9.13}$$

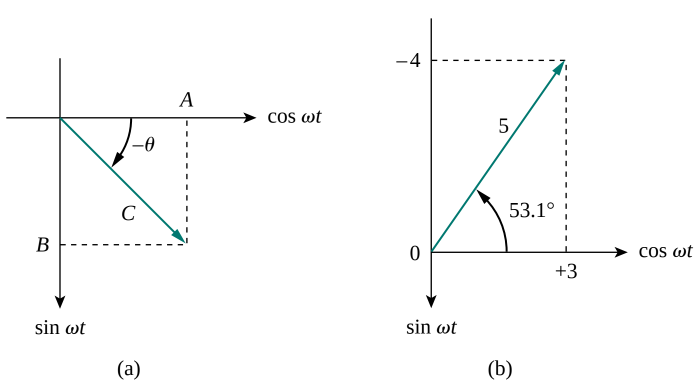

Figure 9.4 (a) Adding $A \cos \omega t$ and $B \sin \omega t$, (b) adding $3 \cos \omega t$ and $-4 \sin \omega t$.

Compared with the trigonometric identities in Eqs. (9.9) and (9.10), the graphical approach eliminates memorization. However, we must not confuse the sine and cosine axes with the axes for complex numbers to be discussed in the next section. Something else to note in Figs. 9.3 and 9.4 is that although the natural tendency is to have the vertical axis point up, the positive direction of the sine function is down in the present case.

EXAMPLE 9.1

Find the amplitude, phase, period, and frequency of the sinusoid

$$v(t) = 12\cos(50t + 10^\circ)$$

Solution:

The amplitude is $V_m = 12$ V.

The phase is $\phi = 10^\circ$.

The angular frequency is $\omega = 50$ rad/s.

The period $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{50} = 0.1257$ s.

The frequency is $f = \dfrac{1}{T} = 7.958$ Hz.

PRACTICE PROBLEM 9.1

Given the sinusoid $5\sin(4\pi t - 60^\circ)$, calculate its amplitude, phase, angular frequency, period, and frequency.

Answer: 5, -60°, 12.57 rad/s, 0.5 s, 2 Hz.

EXAMPLE 9.2

Calculate the phase angle between $v_1 = -10\cos(\omega t + 50^\circ)$ and $v_2 = 12\sin(\omega t - 10^\circ)$. State which sinusoid is leading.

Solution:

Let us calculate the phase in three ways. The first two methods use trigonometric identities, while the third method uses the graphical approach.

METHOD 1 In order to compare v_1 and v_2, we must express them in the same form. If we express them in cosine form with positive amplitudes,

$$v_1 = -10\cos(\omega t + 50^\circ) = 10\cos(\omega t + 50^\circ - 180^\circ)$$

$$v_1 = 10\cos(\omega t - 130^\circ) \quad \text{or} \quad v_1 = 10\cos(\omega t + 230^\circ) \tag{9.2.1}$$

and

$$v_2 = 12\sin(\omega t - 10^\circ) = 12\cos(\omega t - 10^\circ - 90^\circ)$$

$$v_2 = 12\cos(\omega t - 100^\circ) \tag{9.2.2}$$

It can be deduced from Eqs. (9.2.1) and (9.2.2) that the phase difference between v_1 and v_2 is 30°. We can write v_2 as

$$v_2 = 12\cos(\omega t - 130^\circ + 30^\circ) \quad \text{or} \quad v_2 = 12\cos(\omega t + 260^\circ) \tag{9.2.3}$$

Comparing Eqs. (9.2.1) and (9.2.3) shows clearly that v_2 leads v_1 by 30°.

METHOD 2 Alternatively, we may express v_1 in sine form:

$$\begin{aligned} v_1 &= -10\cos(\omega t + 50^\circ) = 10\sin(\omega t + 50^\circ - 90^\circ) \\ &= 10\sin(\omega t - 40^\circ) = 10\sin(\omega t - 10^\circ - 30^\circ) \end{aligned}$$

But $v_2 = 12\sin(\omega t - 10^\circ)$. Comparing the two shows that v_1 lags v_2 by 30°. This is the same as saying that v_2 leads v_1 by 30°.

METHOD 3 We may regard v_1 as simply $-10\cos\omega t$ with a phase shift of $+50^\circ$. Hence, v_1 is as shown in Fig. 9.5. Similarly, v_2 is $12\sin\omega t$ with a phase shift of -10°, as shown in Fig. 9.5. It is easy to see from Fig. 9.5 that v_2 leads v_1 by 30°, that is, $90^\circ - 50^\circ - 10^\circ$.

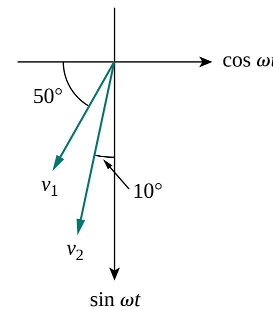

Figure 9.5 For Example 9.2.

PRACTICE PROBLEM 9.2

Find the phase angle between

$$i_1 = -4\sin(377t + 25^\circ) \quad \text{and} \quad i_2 = 5\cos(377t - 40^\circ)$$

Does i_1 lead or lag i_2?

Answer: 155°, i_1 leads i_2.

@ Network Analysis

9.3 PHASORS

Sinusoids are easily expressed in terms of phasors, which are more convenient to work with than sine and cosine functions.

> A *phasor* is a complex number that represents the amplitude and phase of a sinusoid.

Phasors provide a simple means of analyzing linear circuits excited by sinusoidal sources; solutions of such circuits would be intractable otherwise. The notion of solving ac circuits using phasors was first introduced by Charles Steinmetz in 1893. Before we completely define phasors and apply them to circuit analysis, we need to be thoroughly familiar with complex numbers.

Charles Proteus Steinmetz (1865–1923) was a German-Austrian mathematician and electrical engineer.

Appendix B presents a short tutorial on complex numbers.

A complex number z can be written in rectangular form as

$$z = x + jy \tag{9.14a}$$

where $j = \sqrt{-1}$; x is the real part of z; y is the imaginary part of z. In this context, the variables x and y do not represent a location as in two-dimensional vector analysis but rather the real and imaginary parts of z in the complex plane. Nevertheless, we note that there are some resemblances between manipulating complex numbers and manipulating two-dimensional vectors.

The complex number z can also be written in polar or exponential form as

$$z = r\angle\phi = re^{j\phi} \tag{9.14b}$$

where r is the magnitude of z, and ϕ is the phase of z. We notice that z can be represented in three ways:

$$\begin{aligned} z &= x + jy && \text{Rectangular form} \\ z &= r\angle\phi && \text{Polar form} \\ z &= re^{j\phi} && \text{Exponential form} \end{aligned} \tag{9.15}$$

The relationship between the rectangular form and the polar form is shown in Fig. 9.6, where the x axis represents the real part and the y axis represents the imaginary part of a complex number. Given x and y, we can get r and ϕ as

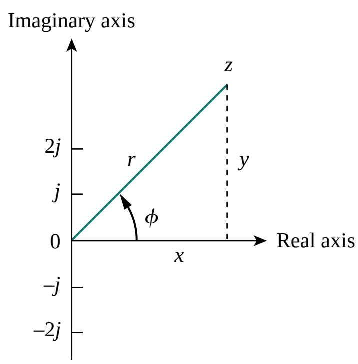

Figure 9.6 Representation of a complex number $z = x + jy = r\angle\phi$.

$$r = \sqrt{x^2 + y^2}, \qquad \phi = \tan^{-1}\frac{y}{x} \tag{9.16a}$$

On the other hand, if we know r and ϕ, we can obtain x and y as

$$x = r\cos\phi, \qquad y = r\sin\phi \tag{9.16b}$$

Thus, z may be written as

$$\boxed{z = x + jy = r\angle\phi = r(\cos\phi + j\sin\phi)} \tag{9.17}$$

Addition and subtraction of complex numbers are better performed in rectangular form; multiplication and division are better done in polar form. Given the complex numbers

$$z = x + jy = r\angle\phi, \qquad z_1 = x_1 + jy_1 = r_1\angle\phi_1$$
$$z_2 = x_2 + jy_2 = r_2\angle\phi_2$$

the following operations are important.

Addition:

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2) \tag{9.18a}$$

Subtraction:

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2) \tag{9.18b}$$

Multiplication:

$$z_1 z_2 = r_1 r_2 \angle\phi_1 + \phi_2 \tag{9.18c}$$

Division:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\angle\phi_1 - \phi_2 \tag{9.18d}$$

Reciprocal:

$$\frac{1}{z} = \frac{1}{r}\angle -\phi \tag{9.18e}$$

Square Root:

$$\sqrt{z} = \sqrt{r}\angle\phi/2 \tag{9.18f}$$

Complex Conjugate:

$$z^* = x - jy = r\underline{/-\phi} = re^{-j\phi} \tag{9.18g}$$

Note that from Eq. (9.18e),

$$\frac{1}{j} = -j \tag{9.18h}$$

These are the basic properties of complex numbers we need. Other properties of complex numbers can be found in Appendix B.

The idea of phasor representation is based on Euler's identity. In general,

$$\boxed{e^{\pm j\phi} = \cos\phi \pm j\sin\phi} \tag{9.19}$$

which shows that we may regard $\cos\phi$ and $\sin\phi$ as the real and imaginary parts of $e^{j\phi}$; we may write

$$\cos\phi = \text{Re}(e^{j\phi}) \tag{9.20a}$$

$$\sin\phi = \text{Im}(e^{j\phi}) \tag{9.20b}$$

where Re and Im stand for the *real part of* and the *imaginary part of*. Given a sinusoid $v(t) = V_m\cos(\omega t + \phi)$, we use Eq. (9.20a) to express $v(t)$ as

$$v(t) = V_m\cos(\omega t + \phi) = \text{Re}(V_m e^{j(\omega t+\phi)}) \tag{9.21}$$

or

$$v(t) = \text{Re}(V_m e^{j\phi} e^{j\omega t}) \tag{9.22}$$

Thus,

$$\boxed{v(t) = \text{Re}(\mathbf{V}e^{j\omega t})} \tag{9.23}$$

where

$$\mathbf{V} = V_m e^{j\phi} = V_m\underline{/\phi} \tag{9.24}$$

$\mathbf{V}$ is thus the *phasor representation* of the sinusoid $v(t)$, as we said earlier. In other words, a phasor is a complex representation of the magnitude and phase of a sinusoid. Either Eq. (9.20a) or Eq. (9.20b) can be used to develop the phasor, but the standard convention is to use Eq. (9.20a).

A phasor may be regarded as a mathematical equivalent of a sinusoid with the time dependence dropped.

One way of looking at Eqs. (9.23) and (9.24) is to consider the plot of the *sinor* $\mathbf{V}e^{j\omega t} = V_m e^{j(\omega t+\phi)}$ on the complex plane. As time increases, the sinor rotates on a circle of radius V_m at an angular velocity ω in the counterclockwise direction, as shown in Fig. 9.7(a). In other words, the entire complex plane is rotating at an angular velocity of ω. We may regard $v(t)$ as the projection of the sinor $\mathbf{V}e^{j\omega t}$ on the real axis, as shown in Fig. 9.7(b). The value of the sinor at time $t = 0$ is the phasor $\mathbf{V}$ of the sinusoid $v(t)$. The sinor may be regarded as a rotating phasor. Thus, whenever a sinusoid is expressed as a phasor, the term $e^{j\omega t}$ is implicitly present. It is therefore important, when dealing with phasors, to keep in mind the frequency ω of the phasor; otherwise we can make serious mistakes.

If we use sine for the phasor instead of cosine, then $v(t) = V_m\sin(\omega t + \phi) = \text{Im}(V_m e^{j(\omega t+\phi)})$ and the corresponding phasor is the same as that in Eq. (9.24).

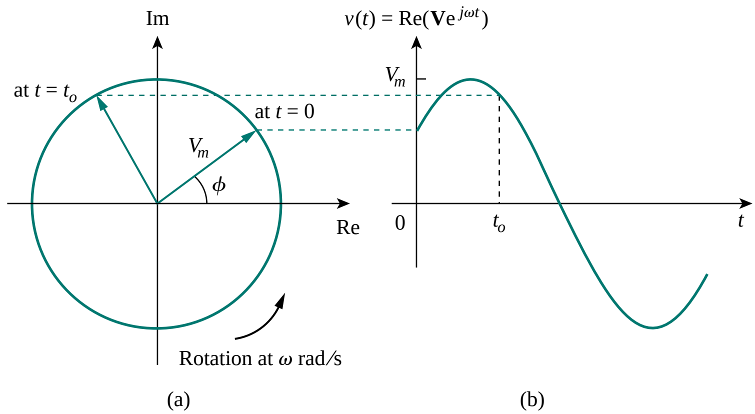

Figure 9.7 Representation of $\mathbf{V}e^{j\omega t}$: (a) sinor rotating counterclockwise, (b) its projection on the real axis, as a function of time.

We use lightface italic letters such as z to represent complex numbers but boldface letters such as **V** to represent phasors, because phasors are vectorlike quantities.

Equation (9.23) states that to obtain the sinusoid corresponding to a given phasor **V**, multiply the phasor by the time factor $e^{j\omega t}$ and take the real part. As a complex quantity, a phasor may be expressed in rectangular form, polar form, or exponential form. Since a phasor has magnitude and phase ("direction"), it behaves as a vector and is printed in boldface. For example, phasors $\mathbf{V} = V_m \underline{/\phi}$ and $\mathbf{I} = I_m \underline{/-\theta}$ are graphically represented in Fig. 9.8. Such a graphical representation of phasors is known as a *phasor diagram.*

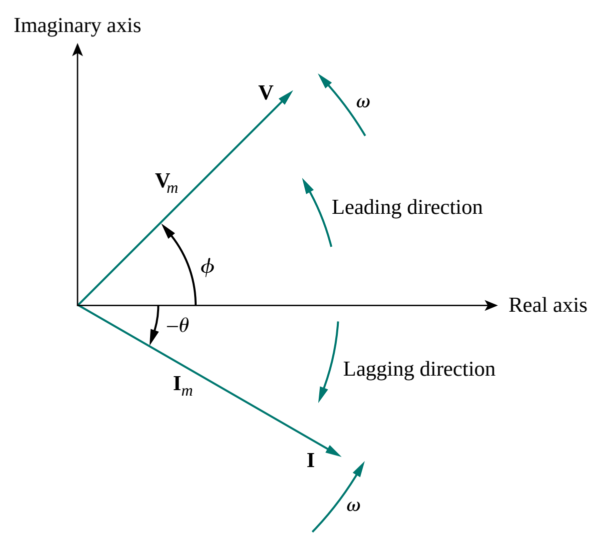

Figure 9.8 A phasor diagram showing $\mathbf{V} = V_m \underline{/\phi}$ and $\mathbf{I} = I_m \underline{/-\theta}$.

Equations (9.21) through (9.23) reveal that to get the phasor corresponding to a sinusoid, we first express the sinusoid in the cosine form so that the sinusoid can be written as the real part of a complex number. Then we take out the time factor $e^{j\omega t}$, andwhatever is left is the phasor corresponding to the sinusoid. By suppressing the time factor, we transform the sinusoid from the time domain to the phasor domain. This transformation is summarized as follows:

$$\underset{\text{(Time-domain representation)}}{v(t) = V_m \cos(\omega t + \phi)} \quad \Longleftrightarrow \quad \underset{\text{(Phasor-domain representation)}}{\mathbf{V} = V_m \angle \phi} \tag{9.25}$$

Given a sinusoid $v(t) = V_m \cos(\omega t + \phi)$, we obtain the corresponding phasor as $\mathbf{V} = V_m \angle \phi$. Equation (9.25) is also demonstrated in Table 9.1, where the sine function is considered in addition to the cosine function. From Eq. (9.25), we see that to get the phasor representation of a sinusoid, we express it in cosine form and take the magnitude and phase. Given a phasor, we obtain the time-domain representation as the cosine function with the same magnitude as the phasor and the argument as ωt plus the phase of the phasor. The idea of expressing information in alternate domains is fundamental to all areas of engineering.

TABLE 9.1 Sinusoid-phasor transformation.

Time-domain representation	Phasor-domain representation
$V_m \cos(\omega t + \phi)$	$V_m \angle \phi$
$V_m \sin(\omega t + \phi)$	$V_m \angle \phi - 90^\circ$
$I_m \cos(\omega t + \theta)$	$I_m \angle \theta$
$I_m \sin(\omega t + \theta)$	$I_m \angle \theta - 90^\circ$

Note that in Eq. (9.25) the frequency (or time) factor $e^{j\omega t}$ is suppressed, and the frequency is not explicitly shown in the phasor-domain representation because ω is constant. However, the response depends on ω. For this reason, the phasor domain is also known as the *frequency domain.*

From Eqs. (9.23) and (9.24), $v(t) = \text{Re}(\mathbf{V}e^{j\omega t}) = V_m \cos(\omega t + \phi)$, so that

$$\begin{aligned}\frac{dv}{dt} &= -\omega V_m \sin(\omega t + \phi) = \omega V_m \cos(\omega t + \phi + 90^\circ) \\ &= \text{Re}(\omega V_m e^{j\omega t} e^{j\phi} e^{j90^\circ}) = \text{Re}(j\omega \mathbf{V} e^{j\omega t})\end{aligned} \tag{9.26}$$

This shows that the derivative $v(t)$ is transformed to the phasor domain as $j\omega\mathbf{V}$

$$\underset{\text{(Time domain)}}{\frac{dv}{dt}} \quad \Longleftrightarrow \quad \underset{\text{(Phasor domain)}}{j\omega\mathbf{V}} \tag{9.27}$$

Differentiating a sinusoid is equivalent to multiplying its corresponding phasor by $j\omega$.

Similarly, the integral of $v(t)$ is transformed to the phasor domain as $\mathbf{V}/j\omega$

$$\underset{\text{(Time domain)}}{\int v\,dt} \quad \Longleftrightarrow \quad \underset{\text{(Phasor domain)}}{\frac{\mathbf{V}}{j\omega}} \tag{9.28}$$

Integrating a sinusoid is equivalent to dividing its corresponding phasor by $j\omega$.

Equation (9.27) allows the replacement of a derivative with respect to time with multiplication of $j\omega$ in the phasor domain, whereas Eq. (9.28) allows the replacement of an integral with respect to time with division by $j\omega$ in the phasor domain. Equations (9.27) and (9.28) are useful in finding the steady-state solution, which does not require knowing the initial values of the variable involved. This is one of the important applications of phasors.

Adding sinusoids of the same frequency is equivalent to adding their corresponding phasors.

Besides time differentiation and integration, another important use of phasors is found in summing sinusoids of the same frequency. This is best illustrated with an example, and Example 9.6 provides one.

The differences between $v(t)$ and $\mathbf{V}$ should be emphasized:

1. $v(t)$ is the *instantaneous or time-domain* representation, while $\mathbf{V}$ is the *frequency or phasor-domain* representation.
2. $v(t)$ is time dependent, while $\mathbf{V}$ is not. (This fact is often forgotten by students.)
3. $v(t)$ is always real with no complex term, while $\mathbf{V}$ is generally complex.

Finally, we should bear in mind that phasor analysis applies only when frequency is constant; it applies in manipulating two or more sinusoidal signals only if they are of the same frequency.

EXAMPLE 9.3

Evaluate these complex numbers:

(a) $(40\angle 50^\circ + 20\angle -30^\circ)^{1/2}$

(b) $\dfrac{10\angle -30^\circ + (3 - j4)}{(2 + j4)(3 - j5)^*}$

Solution:

(a) Using polar to rectangular transformation,

$$40\angle 50^\circ = 40(\cos 50^\circ + j\sin 50^\circ) = 25.71 + j30.64$$
$$20\angle -30^\circ = 20[\cos(-30^\circ) + j\sin(-30^\circ)] = 17.32 - j10$$

Adding them up gives

$$40\angle 50^\circ + 20\angle -30^\circ = 43.03 + j20.64 = 47.72\angle 25.63^\circ$$

Taking the square root of this,

$$(40\angle 50^\circ + 20\angle -30^\circ)^{1/2} = 6.91\angle 12.81^\circ$$

(b) Using polar-rectangular transformation, addition, multiplication, and division,

$$\frac{10\angle -30^\circ + (3 - j4)}{(2 + j4)(3 - j5)^*} = \frac{8.66 - j5 + (3 - j4)}{(2 + j4)(3 + j5)}$$
$$= \frac{11.66 - j9}{-14 + j22} = \frac{14.73\angle -37.66^\circ}{26.08\angle 122.47^\circ}$$
$$= 0.565\angle -160.31^\circ$$

PRACTICE PROBLEM 9.3

Evaluate the following complex numbers:

(a) $[(5 + j2)(-1 + j4) - 5\underline{/60^\circ}]^*$

(b) $\dfrac{10 + j5 + 3\underline{/40^\circ}}{-3 + j4} + 10\underline{/30^\circ}$

Answer: (a) $-15.5 - j13.67$, (b) $8.293 + j2.2$.

EXAMPLE 9.4

Transform these sinusoids to phasors:

(a) $v = -4\sin(30t + 50^\circ)$

(b) $i = 6\cos(50t - 40^\circ)$

Solution:

(a) Since $-\sin A = \cos(A + 90^\circ)$,

$$v = -4\sin(30t + 50^\circ) = 4\cos(30t + 50^\circ + 90^\circ) = 4\cos(30t + 140^\circ)$$

The phasor form of v is

$$\mathbf{V} = 4\underline{/140^\circ}$$

(b) $i = 6\cos(50t - 40^\circ)$ has the phasor

$$\mathbf{I} = 6\underline{/-40^\circ}$$

PRACTICE PROBLEM 9.4

Express these sinusoids as phasors:

(a) $v = -7\cos(2t + 40^\circ)$

(b) $i = 4\sin(10t + 10^\circ)$

Answer: (a) $\mathbf{V} = 7\underline{/220^\circ}$, (b) $\mathbf{I} = 4\underline{/-80^\circ}$.

EXAMPLE 9.5

Find the sinusoids represented by these phasors:

(a) $\mathbf{V} = j8e^{-j20^\circ}$

(b) $\mathbf{I} = -3 + j4$

Solution:

(a) Since $j = 1\underline{/90^\circ}$,

$$\mathbf{V} = j8\underline{/-20^\circ} = (1\underline{/90^\circ})(8\underline{/-20^\circ}) = 8\underline{/90^\circ - 20^\circ} = 8\underline{/70^\circ}\text{ V}$$

Converting this to the time domain gives

$$v(t) = 8\cos(\omega t + 70^\circ)\text{ V}$$

(b) $\mathbf{I} = -3 + j4 = 5\underline{/126.87^\circ}$. Transforming this to the time domain gives

$$i(t) = 5\cos(\omega t + 126.87^\circ)\text{ A}$$

PRACTICE PROBLEM 9.5

Find the sinusoids corresponding to these phasors:

(a) $\mathbf{V} = -10\underline{/30^\circ}$

(b) $\mathbf{I} = j(5 - j12)$

Answer: (a) $v(t) = 10\cos(\omega t + 210^\circ)$, (b) $i(t) = 13\cos(\omega t + 22.62^\circ)$.

EXAMPLE 9.6

Given $i_1(t) = 4\cos(\omega t + 30^\circ)$ and $i_2(t) = 5\sin(\omega t - 20^\circ)$, find their sum.

Solution:

Here is an important use of phasors—for summing sinusoids of the same frequency. Current $i_1(t)$ is in the standard form. Its phasor is

$$\mathbf{I}_1 = 4\underline{/30^\circ}$$

We need to express $i_2(t)$ in cosine form. The rule for converting sine to cosine is to subtract 90°. Hence,

$$i_2 = 5\cos(\omega t - 20^\circ - 90^\circ) = 5\cos(\omega t - 110^\circ)$$

and its phasor is

$$\mathbf{I_2} = 5\underline{/-110^\circ}$$

If we let $i = i_1 + i_2$, then

$$\begin{aligned}\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 &= 4\underline{/30^\circ} + 5\underline{/-110^\circ} \\ &= 3.464 + j2 - 1.71 - j4.698 = 1.754 - j2.698 \\ &= 3.218\underline{/-56.97^\circ}\ \text{A}\end{aligned}$$

Transforming this to the time domain, we get

$$i(t) = 3.218\cos(\omega t - 56.97^\circ)\ \text{A}$$

Of course, we can find $i_1 + i_2$ using Eqs. (9.9), but that is the hard way.

PRACTICE PROBLEM 9.6

If $v_1 = -10\sin(\omega t + 30^\circ)$ and $v_2 = 20\cos(\omega t - 45^\circ)$, find $V = v_1 + v_2$.

Answer: $v(t) = 10.66\cos(\omega t - 30.95^\circ)$.

EXAMPLE 9.7

Using the phasor approach, determine the current $i(t)$ in a circuit described by the integrodifferential equation

$$4i + 8\int i\,dt - 3\frac{di}{dt} = 50\cos(2t + 75^\circ)$$

Solution:

We transform each term in the equation from time domain to phasor domain. Keeping Eqs. (9.27) and (9.28) in mind, we obtain the phasor form of the given equation as

$$4\mathbf{I} + \frac{8\mathbf{I}}{j\omega} - 3j\omega\mathbf{I} = 50\underline{/75^\circ}$$

But $\omega = 2$, so

$$\mathbf{I}(4 - j4 - j6) = 50\underline{/75^\circ}$$

$$\mathbf{I} = \frac{50\underline{/75^\circ}}{4 - j10} = \frac{50\underline{/75^\circ}}{10.77\underline{/-68.2^\circ}} = 4.642\underline{/143.2^\circ} \text{ A}$$

Converting this to the time domain,

$$i(t) = 4.642\cos(2t + 143.2^\circ) \text{ A}$$

Keep in mind that this is only the steady-state solution, and it does not require knowing the initial values.

PRACTICE PROBLEM 9.7

Find the voltage $v(t)$ in a circuit described by the integrodifferential equation

$$2\frac{dv}{dt} + 5v + 10\int v\,dt = 20\cos(5t - 30^\circ)$$

using the phasor approach.

Answer: $v(t) = 2.12\cos(5t - 88^\circ)$.

9.4 PHASOR RELATIONSHIPS FOR CIRCUIT ELEMENTS

Now that we know how to represent a voltage or current in the phasor or frequency domain, one may legitimately ask how we apply this to circuits involving the passive elements R, L, and C. What we need to do is to transform the voltage-current relationship from the time domain to the frequency domain for each element. Again, we will assume the passive sign convention.

We begin with the resistor. If the current through a resistor R is $i = I_m\cos(\omega t + \phi)$, the voltage across it is given by Ohm's law as

$$v = iR = RI_m\cos(\omega t + \phi) \tag{9.29}$$

The phasor form of this voltage is

$$\mathbf{V} = RI_m\underline{/\phi} \tag{9.30}$$

But the phasor representation of the current is $\mathbf{I} = I_m\underline{/\phi}$. Hence,

$$\mathbf{V} = R\mathbf{I} \tag{9.31}$$

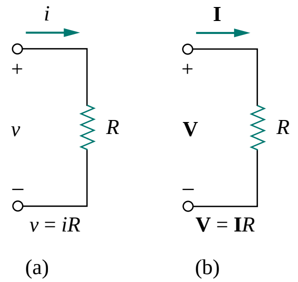

Figure 9.9 Voltage-current relations for a resistor in the: (a) time domain, (b) frequency domain.

showing that the voltage-current relation for the resistor in the phasor domain continues to be Ohm's law, as in the time domain. Figure 9.9 illustrates the voltage-current relations of a resistor. We should note from Eq. (9.31) that voltage and current are in phase, as illustrated in the phasor diagram in Fig. 9.10.

For the inductor L, assume the current through it is $i = I_m \cos(\omega t + \phi)$. The voltage across the inductor is

$$v = L\frac{di}{dt} = -\omega L I_m \sin(\omega t + \phi) \tag{9.32}$$

Recall from Eq. (9.10) that $-\sin A = \cos(A + 90^\circ)$. We can write the voltage as

$$v = \omega L I_m \cos(\omega t + \phi + 90^\circ) \tag{9.33}$$

which transforms to the phasor

$$\mathbf{V} = \omega L I_m e^{j(\phi+90^\circ)} = \omega L I_m e^{j\phi} e^{j90^\circ} = \omega L I_m \underline{/\phi} e^{j90^\circ} \tag{9.34}$$

But $I_m \underline{/\phi} = \mathbf{I}$, and from Eq. (9.19), $e^{j90^\circ} = j$. Thus,

$$\mathbf{V} = j\omega L \mathbf{I} \tag{9.35}$$

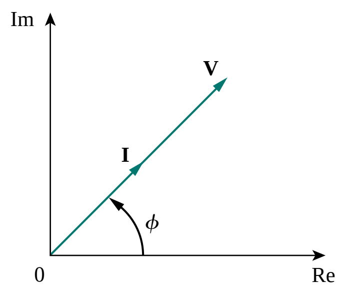

Figure 9.10 Phasor diagram for the resistor.

showing that the voltage has a magnitude of $\omega L I_m$ and a phase of $\phi + 90^\circ$. The voltage and current are 90° out of phase. Specifically, the current lags the voltage by 90°. Figure 9.11 shows the voltage-current relations for the inductor. Figure 9.12 shows the phasor diagram.

Although it is equally correct to say that the inductor voltage leads the current by 90°, convention gives the current phase relative to the voltage.

For the capacitor C, assume the voltage across it is $v = V_m \cos(\omega t + \phi)$. The current through the capacitor is

$$i = C\frac{dv}{dt} \tag{9.36}$$

By following the same steps as we took for the inductor or by applying Eq. (9.27) on Eq. (9.36), we obtain

$$\mathbf{I} = j\omega C \mathbf{V} \quad \Longrightarrow \quad \mathbf{V} = \frac{\mathbf{I}}{j\omega C} \tag{9.37}$$

showing that the current and voltage are 90° out of phase. To be specific, the current leads the voltage by 90°. Figure 9.13 shows the voltage-current

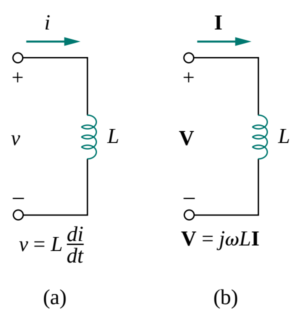

Figure 9.11 Voltage-current relations for an inductor in the: (a) time domain, (b) frequency domain.

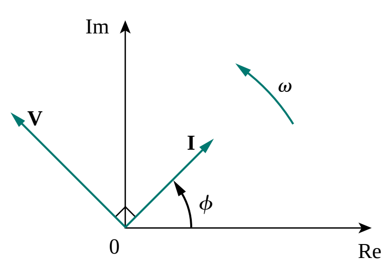

Figure 9.12 Phasor diagram for the inductor; **I** lags **V**.

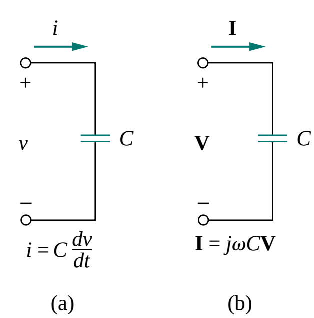

Figure 9.13 Voltage-current relations for a capacitor in the: (a) time domain, (b) frequency domain.

relations for the capacitor; Fig. 9.14 gives the phasor diagram. Table 9.2 summarizes the time-domain and phasor-domain representations of the circuit elements.

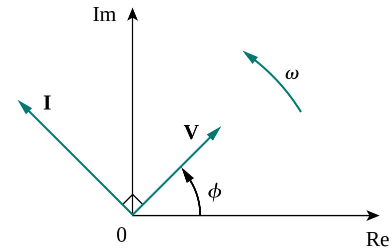

Figure 9.14 Phasor diagram for the capacitor; **I** leads **V**.

TABLE 9.2 Summary of voltage-current relationships.

Element	Time domain	Frequency domain
R	$v = Ri$	$\mathbf{V} = R\mathbf{I}$
L	$v = L\dfrac{di}{dt}$	$\mathbf{V} = j\omega L\mathbf{I}$
C	$i = C\dfrac{dv}{dt}$	$\mathbf{V} = \dfrac{\mathbf{I}}{j\omega C}$

EXAMPLE 9.8

The voltage $v = 12\cos(60t + 45^\circ)$ is applied to a 0.1-H inductor. Find the steady-state current through the inductor.

Solution:

For the inductor, $\mathbf{V} = j\omega L\mathbf{I}$, where $\omega = 60$ rad/s and $\mathbf{V} = 12\angle 45^\circ$ V. Hence

$$\mathbf{I} = \frac{\mathbf{V}}{j\omega L} = \frac{12\angle 45^\circ}{j60 \times 0.1} = \frac{12\angle 45^\circ}{6\angle 90^\circ} = 2\angle -45^\circ \text{ A}$$

Converting this to the time domain,

$$i(t) = 2\cos(60t - 45^\circ) \text{ A}$$

PRACTICE PROBLEM 9.8

If voltage $v = 6\cos(100t - 30^\circ)$ is applied to a 50 μF capacitor, calculate the current through the capacitor.

Answer: $30\cos(100t + 60^\circ)$ mA.

9.5 IMPEDANCE AND ADMITTANCE

In the preceding section, we obtained the voltage-current relations for the three passive elements as

$$\mathbf{V} = R\mathbf{I}, \qquad \mathbf{V} = j\omega L\mathbf{I}, \qquad \mathbf{V} = \frac{\mathbf{I}}{j\omega C} \tag{9.38}$$

These equations may be written in terms of the ratio of the phasor voltage to the phasor current as

$$\frac{\mathbf{V}}{\mathbf{I}} = R, \qquad \frac{\mathbf{V}}{\mathbf{I}} = j\omega L, \qquad \frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{j\omega C} \tag{9.39}$$

From these three expressions, we obtain Ohm's law in phasor form for any type of element as

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} \qquad \text{or} \qquad \mathbf{V} = \mathbf{ZI} \tag{9.40}$$

where $\mathbf{Z}$ is a frequency-dependent quantity known as *impedance*, measured in ohms.

> The **impedance Z** of a circuit is the ratio of the phasor voltage **V** to the phasor current **I**, measured in ohms (Ω).

The impedance represents the opposition which the circuit exhibits to the flow of sinusoidal current. Although the impedance is the ratio of two phasors, it is not a phasor, because it does not correspond to a sinusoidally varying quantity.

The impedances of resistors, inductors, and capacitors can be readily obtained from Eq. (9.39). Table 9.3 summarizes their impedances and admittance. From the table we notice that $\mathbf{Z}_L = j\omega L$ and $\mathbf{Z}_C = -j/\omega C$. Consider two extreme cases of angular frequency. When $\omega = 0$ (i.e., for dc sources), $\mathbf{Z}_L = 0$ and $\mathbf{Z}_C \rightarrow \infty$, confirming what we already know—that the inductor acts like a short circuit, while the capacitor acts like an open circuit. When $\omega \rightarrow \infty$ (i.e., for high frequencies), $\mathbf{Z}_L \rightarrow \infty$ and $\mathbf{Z}_C = 0$, indicating that the inductor is an open circuit to high frequencies, while the capacitor is a short circuit. Figure 9.15 illustrates this.

TABLE 9.3 Impedances and admittances of passive elements.

Element	Impedance	Admittance
R	$\mathbf{Z} = R$	$\mathbf{Y} = \dfrac{1}{R}$
L	$\mathbf{Z} = j\omega L$	$\mathbf{Y} = \dfrac{1}{j\omega L}$
C	$\mathbf{Z} = \dfrac{1}{j\omega C}$	$\mathbf{Y} = j\omega C$

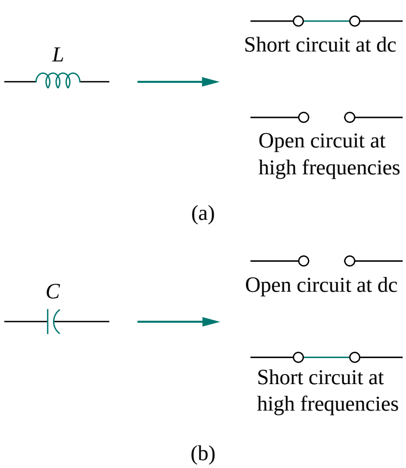

Figure 9.15 Equivalent circuits at dc and high frequencies: (a) inductor, (b) capacitor.

As a complex quantity, the impedance may be expressed in rectangular form as

$$\mathbf{Z} = R + jX \tag{9.41}$$

where $R = \text{Re}\,\mathbf{Z}$ is the *resistance* and $X = \text{Im}\,\mathbf{Z}$ is the *reactance*. The reactance X may be positive or negative. We say that the impedance is inductive when X is positive or capacitive when X is negative. Thus, impedance $\mathbf{Z} = R + jX$ is said to be *inductive* or lagging since current lags voltage, while impedance $\mathbf{Z} = R - jX$ is capacitive or leading because current leads voltage. The impedance, resistance, and reactance are all measured in ohms. The impedance may also be expressed in polar form as

$$\mathbf{Z} = |\mathbf{Z}|\underline{/\theta} \tag{9.42}$$

Comparing Eqs. (9.41) and (9.42), we infer that

$$\mathbf{Z} = R + jX = |\mathbf{Z}|\underline{/\theta} \tag{9.43}$$

where

$$|\mathbf{Z}| = \sqrt{R^2 + X^2}, \qquad \theta = \tan^{-1}\frac{X}{R} \tag{9.44}$$

and

$$R = |\mathbf{Z}|\cos\theta, \qquad X = |\mathbf{Z}|\sin\theta \tag{9.45}$$

It is sometimes convenient to work with the reciprocal of impedance, known as *admittance*.

The **admittance** **Y** is the reciprocal of impedance, measured in siemens (S).

The admittance **Y** of an element (or a circuit) is the ratio of the phasor current through it to the phasor voltage across it, or

$$\mathbf{Y} = \frac{1}{\mathbf{Z}} = \frac{\mathbf{I}}{\mathbf{V}} \tag{9.46}$$

The admittances of resistors, inductors, and capacitors can be obtained from Eq. (9.39). They are also summarized in Table 9.3.

As a complex quantity, we may write **Y** as

$$\mathbf{Y} = G + jB \tag{9.47}$$

where $G = \text{Re}\,\mathbf{Y}$ is called the *conductance* and $B = \text{Im}\,\mathbf{Y}$ is called the *susceptance*. Admittance, conductance, and susceptance are all expressed in the unit of siemens (or mhos). From Eqs. (9.41) and (9.47),

$$G + jB = \frac{1}{R + jX} \tag{9.48}$$

By rationalization,

$$G + jB = \frac{1}{R + jX} \cdot \frac{R - jX}{R - jX} = \frac{R - jX}{R^2 + X^2} \tag{9.49}$$

Equating the real and imaginary parts gives

$$G = \frac{R}{R^2 + X^2}, \qquad B = -\frac{X}{R^2 + X^2} \tag{9.50}$$

showing that $G \neq 1/R$ as it is in resistive circuits. Of course, if $X = 0$, then $G = 1/R$.

EXAMPLE 9.9

Find $v(t)$ and $i(t)$ in the circuit shown in Fig. 9.16.

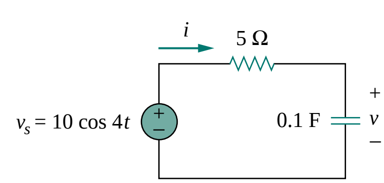

Figure 9.16 For Example 9.9.

Solution:

From the voltage source $10 \cos 4t$, $\omega = 4$,

$$\mathbf{V}_s = 10\underline{/0^\circ} \text{ V}$$

The impedance is

$$\mathbf{Z} = 5 + \frac{1}{j\omega C} = 5 + \frac{1}{j4 \times 0.1} = 5 - j2.5\ \Omega$$

Hence the current

$$\begin{aligned}\mathbf{I} = \frac{\mathbf{V}_s}{\mathbf{Z}} &= \frac{10\underline{/0^\circ}}{5 - j2.5} = \frac{10(5 + j2.5)}{5^2 + 2.5^2} \\ &= 1.6 + j0.8 = 1.789\underline{/26.57^\circ} \text{ A}\end{aligned} \tag{9.9.1}$$

The voltage across the capacitor is

$$\mathbf{V} = \mathbf{I}\mathbf{Z}_C = \frac{\mathbf{I}}{j\omega C} = \frac{1.789\underline{/26.57^\circ}}{j4 \times 0.1} = \frac{1.789\underline{/26.57^\circ}}{0.4\underline{/90^\circ}} = 4.47\underline{/-63.43^\circ} \text{ V} \tag{9.9.2}$$

Converting **I** and **V** in Eqs. (9.9.1) and (9.9.2) to the time domain, we get

$$i(t) = 1.789\cos(4t + 26.57^\circ) \text{ A}$$
$$v(t) = 4.47\cos(4t - 63.43^\circ) \text{ V}$$

Notice that $i(t)$ leads $v(t)$ by 90° as expected.

PRACTICE PROBLEM 9.9

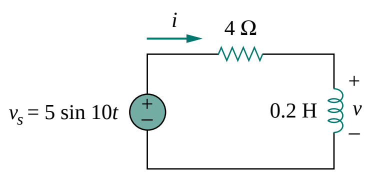

Figure 9.17 For Practice Prob. 9.9.

Refer to Fig. 9.17. Determine $v(t)$ and $i(t)$.

Answer: $2.236\sin(10t + 63.43^\circ)$ V, $1.118\sin(10t - 26.57^\circ)$ A.

†9.6 KIRCHHOFF'S LAWS IN THE FREQUENCY DOMAIN

We cannot do circuit analysis in the frequency domain without Kirchhoff's current and voltage laws. Therefore, we need to express them in the frequency domain.

For KVL, let $v_1, v_2, \ldots, v_n$ be the voltages around a closed loop. Then

$$v_1 + v_2 + \cdots + v_n = 0 \tag{9.51}$$

In the sinusoidal steady state, each voltage may be written in cosine form, so that Eq. (9.51) becomes

$$V_{m1}\cos(\omega t + \theta_1) + V_{m2}\cos(\omega t + \theta_2) + \cdots + V_{mn}\cos(\omega t + \theta_n) = 0 \tag{9.52}$$

This can be written as

$$\text{Re}(V_{m1}e^{j\theta_1}e^{j\omega t}) + \text{Re}(V_{m2}e^{j\theta_2}e^{j\omega t}) + \cdots + \text{Re}(V_{mn}e^{j\theta_n}e^{j\omega t}) = 0$$

or

$$\text{Re}[(V_{m1}e^{j\theta_1} + V_{m2}e^{j\theta_2} + \cdots + V_{mn}e^{j\theta_n})e^{j\omega t}] = 0 \tag{9.53}$$

If we let $\mathbf{V}_k = V_{mk}e^{j\theta_k}$, then

$$\text{Re}[(\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n)e^{j\omega t}] = 0 \tag{9.54}$$

Since $e^{j\omega t} \neq 0$,

$$\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n = 0 \tag{9.55}$$

indicating that Kirchhoff's voltage law holds for phasors.

By following a similar procedure, we can show that Kirchhoff's current law holds for phasors. If we let $i_1, i_2, \dots, i_n$ be the current leaving or entering a closed surface in a network at time t, then

$$i_1 + i_2 + \cdots + i_n = 0 \qquad (9.56)$$

If $\mathbf{I}_1, \mathbf{I}_2, \dots, \mathbf{I}_n$ are the phasor forms of the sinusoids $i_1, i_2, \dots, i_n$, then

$$\mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_n = 0 \qquad (9.57)$$

which is Kirchhoff's current law in the frequency domain.

Once we have shown that both KVL and KCL hold in the frequency domain, it is easy to do many things, such as impedance combination, nodal and mesh analyses, superposition, and source transformation.

9.7 IMPEDANCE COMBINATIONS

Consider the N series-connected impedances shown in Fig. 9.18. The same current $\mathbf{I}$ flows through the impedances. Applying KVL around the loop gives

$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_N = \mathbf{I}(\mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N) \qquad (9.58)$$

The equivalent impedance at the input terminals is

$$\mathbf{Z}_{eq} = \frac{\mathbf{V}}{\mathbf{I}} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N$$

or

$$\boxed{\mathbf{Z}_{eq} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N} \qquad (9.59)$$

showing that the total or equivalent impedance of series-connected impedances is the sum of the individual impedances. This is similar to the series connection of resistances.

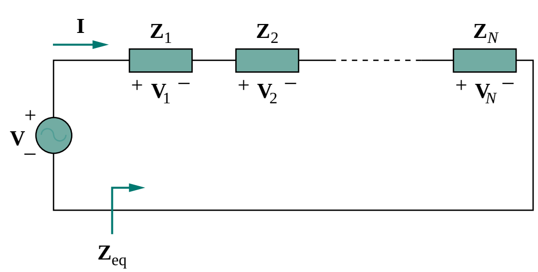

Figure 9.18 N impedances in series.

If $N = 2$, as shown in Fig. 9.19, the current through the impedances is

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}_1 + \mathbf{Z}_2} \qquad (9.60)$$

Since $\mathbf{V}_1 = \mathbf{Z}_1\mathbf{I}$ and $\mathbf{V}_2 = \mathbf{Z}_2\mathbf{I}$, then

$$\boxed{\mathbf{V}_1 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}, \qquad \mathbf{V}_2 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}} \qquad (9.61)$$

which is the *voltage-division* relationship.

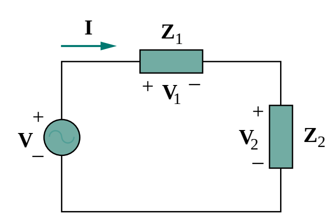

Figure 9.19 Voltage division.

In the same manner, we can obtain the equivalent impedance or admittance of the N parallel-connected impedances shown in Fig. 9.20. The voltage across each impedance is the same. Applying KCL at the top node,

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_N = \mathbf{V}\left(\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_N}\right) \tag{9.62}$$

The equivalent impedance is

$$\frac{1}{\mathbf{Z}_{\mathrm{eq}}} = \frac{\mathbf{I}}{\mathbf{V}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_N} \tag{9.63}$$

and the equivalent admittance is

$$\boxed{\mathbf{Y}_{\mathrm{eq}} = \mathbf{Y}_1 + \mathbf{Y}_2 + \cdots + \mathbf{Y}_N} \tag{9.64}$$

This indicates that the equivalent admittance of a parallel connection of admittances is the sum of the individual admittances.

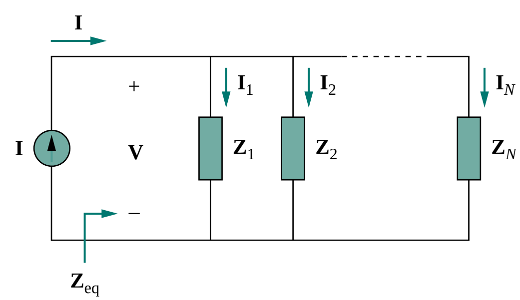

Figure 9.20 N impedances in parallel.

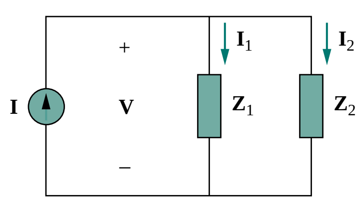

Figure 9.21 Current division.

When $N = 2$, as shown in Fig. 9.21, the equivalent impedance becomes

$$\mathbf{Z}_{\mathrm{eq}} = \frac{1}{\mathbf{Y}_{\mathrm{eq}}} = \frac{1}{\mathbf{Y}_1 + \mathbf{Y}_2} = \frac{1}{1/\mathbf{Z}_1 + 1/\mathbf{Z}_2} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2} \tag{9.65}$$

Also, since

$$\mathbf{V} = \mathbf{I}\mathbf{Z}_{\mathrm{eq}} = \mathbf{I}_1\mathbf{Z}_1 = \mathbf{I}_2\mathbf{Z}_2$$

the currents in the impedances are

$$\boxed{\mathbf{I}_1 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I}, \qquad \mathbf{I}_2 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I}} \tag{9.66}$$

which is the *current-division* principle.

The delta-to-wye and wye-to-delta transformations that we applied to resistive circuits are also valid for impedances. With reference to Fig. 9.22, the conversion formulas are as follows.

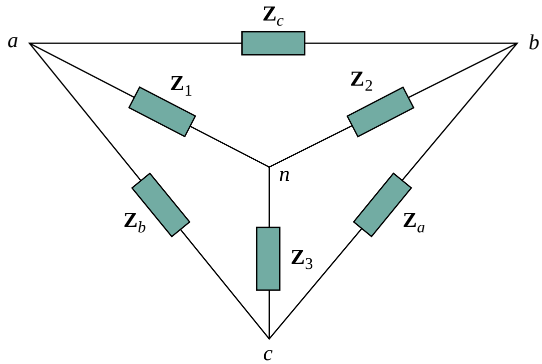

Figure 9.22 Superimposed Y and Δ networks.

Y-Δ Conversion:

$$\mathbf{Z}_a = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_1}$$
$$\mathbf{Z}_b = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_2} \tag{9.67}$$
$$\mathbf{Z}_c = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_3}$$

Δ-Y Conversion:

$$\mathbf{Z}_1 = \frac{\mathbf{Z}_b\mathbf{Z}_c}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c}$$
$$\mathbf{Z}_2 = \frac{\mathbf{Z}_c\mathbf{Z}_a}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c} \tag{9.68}$$
$$\mathbf{Z}_3 = \frac{\mathbf{Z}_a\mathbf{Z}_b}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c}$$

A delta or wye circuit is said to be balanced if it has equal impedances in all three branches.

When a Δ-Y circuit is balanced, Eqs. (9.67) and (9.68) become

$$\mathbf{Z}_\Delta = 3\mathbf{Z}_Y \qquad \text{or} \qquad \mathbf{Z}_Y = \frac{1}{3}\mathbf{Z}_\Delta \tag{9.69}$$

where $\mathbf{Z}_Y = \mathbf{Z}_1 = \mathbf{Z}_2 = \mathbf{Z}_3$ and $\mathbf{Z}_\Delta = \mathbf{Z}_a = \mathbf{Z}_b = \mathbf{Z}_c$.

As you see in this section, the principles of voltage division, current division, circuit reduction, impedance equivalence, and Y-Δ transformation all apply to ac circuits. Chapter 10 will show that other circuit techniques—such as superposition, nodal analysis, mesh analysis, source transformation, the Thevenin theorem, and the Norton theorem—are all applied to ac circuits in a manner similar to their application in dc circuits.

EXAMPLE 9.10

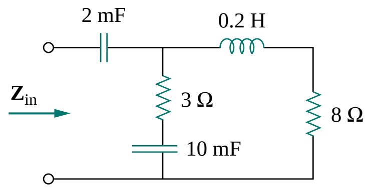

Figure 9.23 For Example 9.10.

Find the input impedance of the circuit in Fig. 9.23. Assume that the circuit operates at $\omega = 50$ rad/s.

Solution:

Let

$\mathbf{Z}_1$ = Impedance of the 2-mF capacitor

$\mathbf{Z}_2$ = Impedance of the 3-Ω resistor in series with the 10-mF capacitor

$\mathbf{Z}_3$ = Impedance of the 0.2-H inductor in series with the 8-Ω resistor

Then

$$\mathbf{Z}_1 = \frac{1}{j\omega C} = \frac{1}{j50 \times 2 \times 10^{-3}} = -j10\ \Omega$$

$$\mathbf{Z}_2 = 3 + \frac{1}{j\omega C} = 3 + \frac{1}{j50 \times 10 \times 10^{-3}} = (3 - j2)\ \Omega$$

$$\mathbf{Z}_3 = 8 + j\omega L = 8 + j50 \times 0.2 = (8 + j10)\ \Omega$$

The input impedance is

$$\mathbf{Z}_{in} = \mathbf{Z}_1 + \mathbf{Z}_2 \parallel \mathbf{Z}_3 = -j10 + \frac{(3 - j2)(8 + j10)}{11 + j8}$$

$$= -j10 + \frac{(44 + j14)(11 - j8)}{11^2 + 8^2} = -j10 + 3.22 - j1.07\ \Omega$$

Thus,

$$\mathbf{Z}_{in} = 3.22 - j11.07\ \Omega$$

PRACTICE PROBLEM 9.10

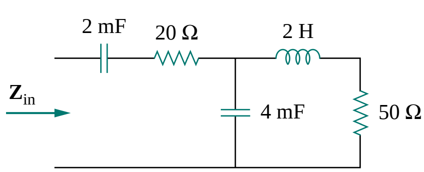

Figure 9.24 For Practice Prob. 9.10.

Determine the input impedance of the circuit in Fig. 9.24 at $\omega = 10$ rad/s.

Answer: $32.38 - j73.76\ \Omega$.

EXAMPLE 9.11

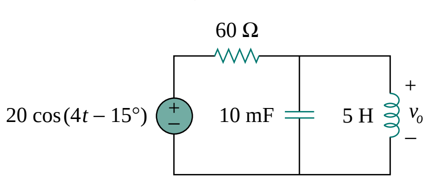

Figure 9.25 For Example 9.11.

Determine $v_o(t)$ in the circuit in Fig. 9.25.

Solution:

To do the analysis in the frequency domain, we must first transform the time-domain circuit in Fig. 9.25 to the phasor-domain equivalent in Fig. 9.26. The transformation produces

$$v_s = 20\cos(4t - 15^\circ) \quad \Longrightarrow \quad \mathbf{V}_s = 20\underline{/-15^\circ}\text{ V}, \qquad \omega = 4$$

$$10\text{ mF} \quad \Longrightarrow \quad \frac{1}{j\omega C} = \frac{1}{j4 \times 10 \times 10^{-3}} = -j25\ \Omega$$

$$5\text{ H} \quad \Longrightarrow \quad j\omega L = j4 \times 5 = j20\ \Omega$$

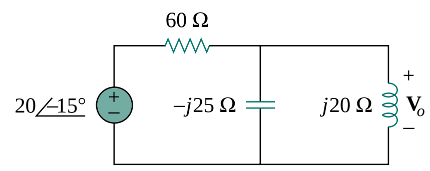

Figure 9.26 The frequency-domain equivalent of the circuit in Fig. 9.25.

Let

$\mathbf{Z}_1$ = Impedance of the 60-Ω resistor

$\mathbf{Z}_2$ = Impedance of the parallel combination of the 10-mF capacitor and the 5-H inductor

Then $\mathbf{Z}_1 = 60\ \Omega$ and

$$\mathbf{Z}_2 = -j25 \parallel j20 = \frac{-j25 \times j20}{-j25 + j20} = j100\ \Omega$$

By the voltage-division principle,

$$\mathbf{V}_o = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}_s = \frac{j100}{60 + j100}(20\underline{/-15^\circ})$$

$$= (0.8575\underline{/30.96^\circ})(20\underline{/-15^\circ}) = 17.15\underline{/15.96^\circ}\text{ V.}$$

We convert this to the time domain and obtain

$$v_o(t) = 17.15\cos(4t + 15.96^\circ)\text{V}$$

PRACTICE PROBLEM 9.11

Calculate v_o in the circuit in Fig. 9.27.

Answer: $v_o(t) = 7.071\cos(10t - 60^\circ)$ V.

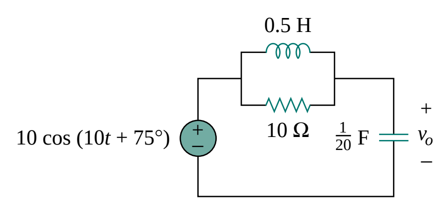

Figure 9.27 For Practice Prob. 9.11.

EXAMPLE 9.12

Find current **I** in the circuit in Fig. 9.28.

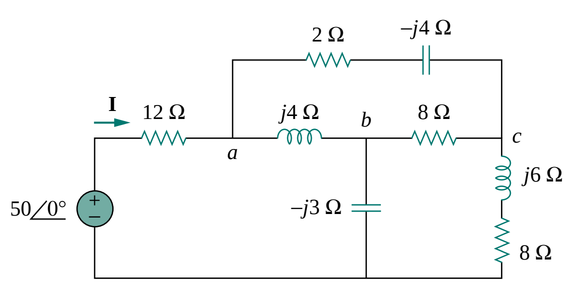

Figure 9.28 For Example 9.12.

Solution:

The delta network connected to nodes a, b, and c can be converted to the Y network of Fig. 9.29. We obtain the Y impedances as follows using Eq. (9.68):

$$\mathbf{Z}_{an} = \frac{j4(2 - j4)}{j4 + 2 - j4 + 8} = \frac{4(4 + j2)}{10} = (1.6 + j0.8)\ \Omega$$

$$\mathbf{Z}_{bn} = \frac{j4(8)}{10} = j3.2\ \Omega, \qquad \mathbf{Z}_{cn} = \frac{8(2 - j4)}{10} = (1.6 - j3.2)\ \Omega$$

The total impedance at the source terminals is

$$\begin{aligned}\mathbf{Z} &= 12 + \mathbf{Z}_{an} + (\mathbf{Z}_{bn} - j3) \parallel (\mathbf{Z}_{cn} + j6 + 8)\\ &= 12 + 1.6 + j0.8 + (j0.2) \parallel (9.6 + j2.8)\\ &= 13.6 + j0.8 + \frac{j0.2(9.6 + j2.8)}{9.6 + j3}\\ &= 13.6 + j1 = 13.64\underline{/4.204^\circ}\ \Omega\end{aligned}$$

The desired current is

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{50\underline{/0^\circ}}{13.64\underline{/4.204^\circ}} = 3.666\underline{/-4.204^\circ}\ \text{A}$$

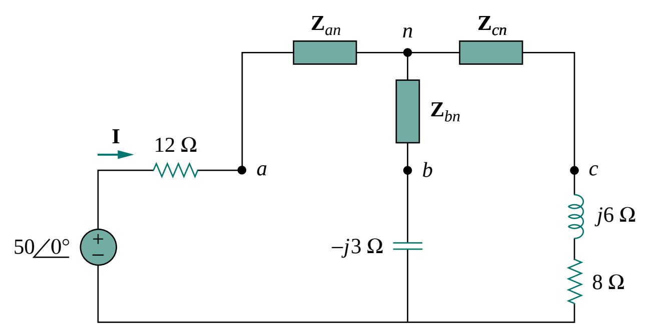

Figure 9.29 The circuit in Fig. 9.28 after delta-to-wye transformation.

PRACTICE PROBLEM 9.12

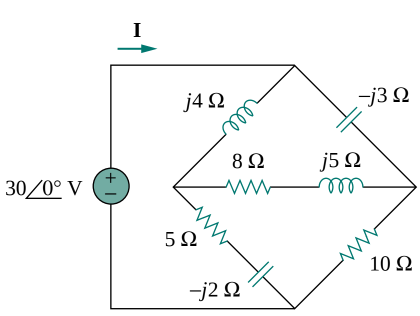

Figure 9.30 For Practice Prob. 9.12.

Find **I** in the circuit in Fig. 9.30.

Answer: $6.364\underline{/3.802^\circ}$ A.

†9.8 APPLICATIONS

In Chapters 7 and 8, we saw certain uses of RC, RL, and RLC circuits in dc applications. These circuits also have ac applications; among them are coupling circuits, phase-shifting circuits, filters, resonant circuits, ac bridge circuits, and transformers. This list of applications is inexhaustive. We will consider some of them later. It will suffice here to observe two simple ones: RC phase-shifting circuits, and ac bridge circuits.

9.8.1 Phase-Shifters

A phase-shifting circuit is often employed to correct an undesirable phase shift already present in a circuit or to produce special desired effects. An RC circuit is suitable for this purpose because its capacitor causes the circuit current to lead the applied voltage. Two commonly used RC circuits are shown in Fig. 9.31. (RL circuits or any reactive circuits could also serve the same purpose.)

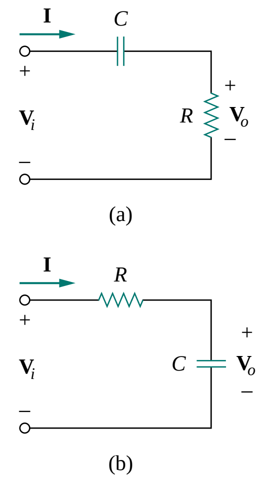

Figure 9.31 Series RC shift circuits: (a) leading output, (b) lagging output.

In Fig. 9.31(a), the circuit current $\mathbf{I}$ leads the applied voltage $\mathbf{V}_i$ by some phase angle θ, where $0 < \theta < 90^\circ$, depending on the values of R and C. If $X_C = -1/\omega C$, then the total impedance is $\mathbf{Z} = R + jX_C$, and the phase shift is given by

$$\theta = \tan^{-1} \frac{X_C}{R} \tag{9.70}$$

This shows that the amount of phase shift depends on the values of R, C, and the operating frequency. Since the output voltage $\mathbf{V}_o$ across the resistor is in phase with the current, $\mathbf{V}_o$ leads (positive phase shift) $\mathbf{V}_i$ as shown in Fig. 9.32(a).

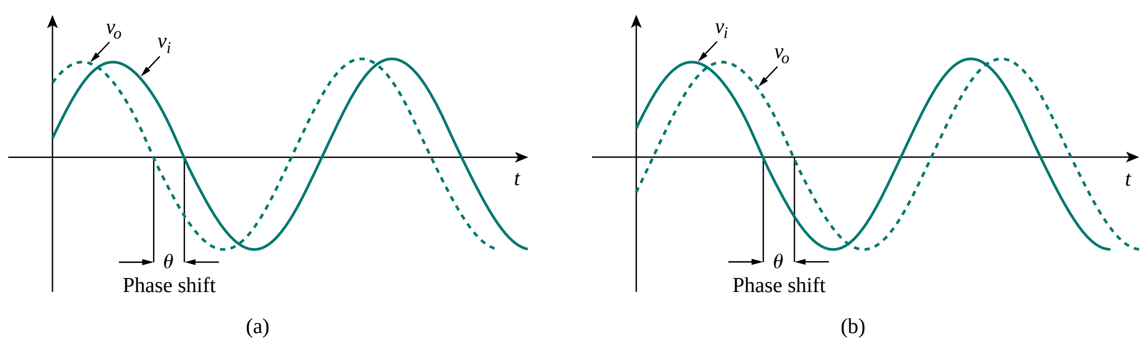

Figure 9.32 Phase shift in RC circuits: (a) leading output, (b) lagging output.

In Fig. 9.31(b), the output is taken across the capacitor. The current $\mathbf{I}$ leads the input voltage $\mathbf{V}_i$ by θ, but the output voltage $v_o(t)$ across the capacitor lags (negative phase shift) the input voltage $v_i(t)$ as illustrated in Fig. 9.32(b).

We should keep in mind that the simple RC circuits in Fig. 9.31 also act as voltage dividers. Therefore, as the phase shift θ approaches 90°, the output voltage $\mathbf{V}_o$ approaches zero. For this reason, these simple RC circuits are used only when small amounts of phase shift are required.

If it is desired to have phase shifts greater than 60°, simple RC networks are cascaded, thereby providing a total phase shift equal to the sum of the individual phase shifts. In practice, the phase shifts due to the stages are not equal, because the succeeding stages load down the earlier stages unless op amps are used to separate the stages.

EXAMPLE 9.13

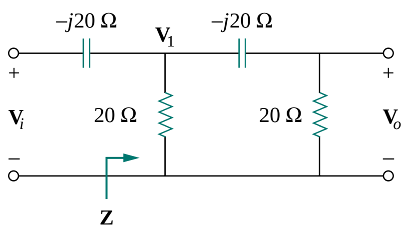

Figure 9.33 An RC phase shift circuit with 90° leading phase shift; for Example 9.13.

Design an RC circuit to provide a phase of 90° leading.

Solution:

If we select circuit components of equal ohmic value, say $R = |X_C| = 20\ \Omega$, at a particular frequency, according to Eq. (9.70), the phase shift is exactly 45°. By cascading two similar RC circuits in Fig. 9.31(a), we obtain the circuit in Fig. 9.33, providing a positive or leading phase shift of 90°, as we shall soon show. Using the series-parallel combination technique, **Z** in Fig. 9.33 is obtained as

$$\mathbf{Z} = 20 \parallel (20 - j20) = \frac{20(20 - j20)}{40 - j20} = 12 - j4\ \Omega \qquad (9.13.1)$$

Using voltage division,

$$\mathbf{V}_1 = \frac{\mathbf{Z}}{\mathbf{Z} - j20}\mathbf{V}_i = \frac{12 - j4}{12 - j24}\mathbf{V}_i = \frac{\sqrt{2}}{3}\angle 45^\circ \mathbf{V}_i \qquad (9.13.2)$$

and

$$\mathbf{V}_o = \frac{20}{20 - j20}\mathbf{V}_1 = \frac{\sqrt{2}}{2}\angle 45^\circ \mathbf{V}_1 \qquad (9.13.3)$$

Substituting Eq. (9.13.2) into Eq. (9.13.3) yields

$$\mathbf{V}_o = \left(\frac{\sqrt{2}}{2}\angle 45^\circ\right)\left(\frac{\sqrt{2}}{3}\angle 45^\circ \mathbf{V}_i\right) = \frac{1}{3}\angle 90^\circ \mathbf{V}_i$$

Thus, the output leads the input by 90° but its magnitude is only about 33 percent of the input.

PRACTICE PROBLEM 9.13

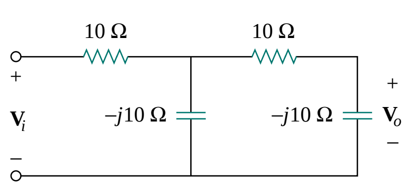

Figure 9.34 For Practice Prob. 9.13.

Design an RC circuit to provide a 90° lagging phase shift. If a voltage of 10 V is applied, what is the output voltage?

Answer: Figure 9.34 shows a typical design; 3.33 V.

EXAMPLE 9.14

For the RL circuit shown in Fig. 9.35(a), calculate the amount of phase shift produced at 2 kHz.

Solution:

At 2 kHz, we transform the 10-mH and 5-mH inductances to the corresponding impedances.

$$10 \text{ mH} \quad \Longrightarrow \quad X_L = \omega L = 2\pi \times 2 \times 10^3 \times 10 \times 10^{-3} = 40\pi = 125.7\ \Omega$$

$$5 \text{ mH} \quad \Longrightarrow \quad X_L = \omega L = 2\pi \times 2 \times 10^3 \times 5 \times 10^{-3} = 20\pi = 62.83\ \Omega$$

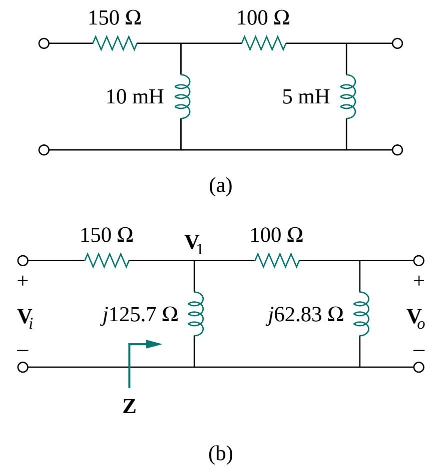

Figure 9.35 For Example 9.14.

Consider the circuit in Fig. 9.35(b). The impedance **Z** is the parallel combination of $j125.7\ \Omega$ and $100 + j62.83\ \Omega$. Hence,

$$\mathbf{Z} = j125.7 \parallel (100 + j62.83) = \frac{j125.7(100 + j62.83)}{100 + j188.5} = 69.56\angle 60.1^\circ\ \Omega \tag{9.14.1}$$

Using voltage division,

$$\mathbf{V}_1 = \frac{\mathbf{Z}}{\mathbf{Z} + 150}\mathbf{V}_i = \frac{69.56\angle 60.1^\circ}{184.7 + j60.3}\mathbf{V}_i = 0.3582\angle 42.02^\circ\ \mathbf{V}_i \tag{9.14.2}$$

and

$$\mathbf{V}_o = \frac{j62.832}{100 + j62.832}\mathbf{V}_1 = 0.532\angle 57.86^\circ\ \mathbf{V}_1 \tag{9.14.3}$$

Combining Eqs. (9.14.2) and (9.14.3),

$$\mathbf{V}_o = (0.532\angle 57.86^\circ)(0.3582\angle 42.02^\circ)\ \mathbf{V}_i = 0.1906\angle 100^\circ\ \mathbf{V}_i$$

showing that the output is about 19 percent of the input in magnitude but leading the input by 100°. If the circuit is terminated by a load, the load will affect the phase shift.

PRACTICE PROBLEM 9.14

Refer to the RL circuit in Fig. 9.36. If 1 V is applied, find the magnitude and the phase shift produced at 5 kHz. Specify whether the phase shift is leading or lagging.

Answer: 0.172, 120.4°, lagging.

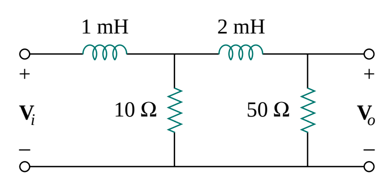

Figure 9.36 For Practice Prob. 9.14.

9.8.2 AC Bridges

An ac bridge circuit is used in measuring the inductance L of an inductor or the capacitance C of a capacitor. It is similar in form to the Wheatstone bridge for measuring an unknown resistance (discussed in Section 4.10) and follows the same principle. To measure L and C, however, an ac source is needed as well as an ac meter instead of the galvanometer. The ac meter may be a sensitive ac ammeter or voltmeter.

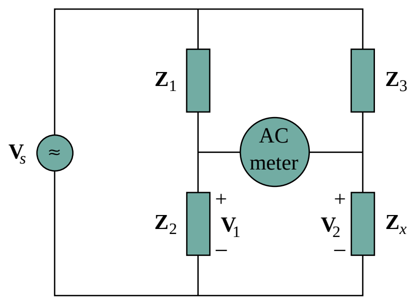

Figure 9.37 A general ac bridge.

Consider the general ac bridge circuit displayed in Fig. 9.37. The bridge is *balanced* when no current flows through the meter. This means that $\mathbf{V}_1 = \mathbf{V}_2$. Applying the voltage division principle,

$$\mathbf{V}_1 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}_s = \mathbf{V}_2 = \frac{\mathbf{Z}_x}{\mathbf{Z}_3 + \mathbf{Z}_x}\mathbf{V}_s \tag{9.71}$$

Thus,

$$\frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2} = \frac{\mathbf{Z}_x}{\mathbf{Z}_3 + \mathbf{Z}_x} \quad \Longrightarrow \quad \mathbf{Z}_2\mathbf{Z}_3 = \mathbf{Z}_1\mathbf{Z}_x \tag{9.72}$$

or

$$\boxed{\mathbf{Z}_x = \frac{\mathbf{Z}_3}{\mathbf{Z}_1}\mathbf{Z}_2} \tag{9.73}$$

This is the balanced equation for the ac bridge and is similar to Eq. (4.30) for the resistance bridge except that the R's are replaced by $\mathbf{Z}$'s.

Specific ac bridges for measuring L and C are shown in Fig. 9.38, where L_x and C_x are the unknown inductance and capacitance to be measured while L_s and C_s are a standard inductance and capacitance (the values of which are known to great precision). In each case, two resistors, R_1 and R_2, are varied until the ac meter reads zero. Then the bridge is balanced. From Eq. (9.73), we obtain

$$L_x = \frac{R_2}{R_1}L_s \tag{9.74}$$

and

$$C_x = \frac{R_1}{R_2}C_s \tag{9.75}$$

Notice that the balancing of the ac bridges in Fig. 9.38 does not depend on the frequency f of the ac source, since f does not appear in the relationships in Eqs. (9.74) and (9.75).

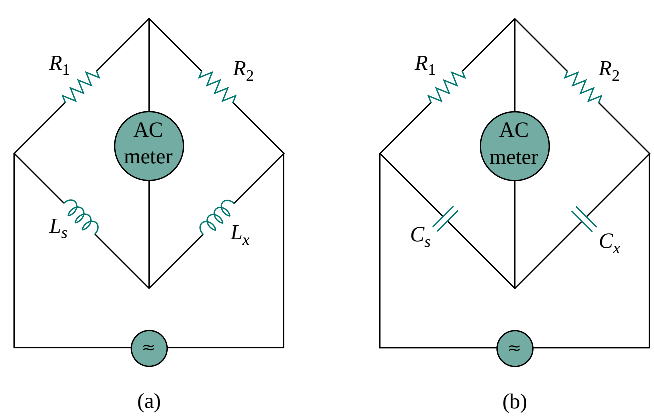

Figure 9.38 Specific ac bridges: (a) for measuring L, (b) for measuring C.

EXAMPLE 9.15

The ac bridge circuit of Fig. 9.37 balances when $\mathbf{Z}_1$ is a 1-kΩ resistor, $\mathbf{Z}_2$ is a 4.2-kΩ resistor, $\mathbf{Z}_3$ is a parallel combination of a 1.5-MΩ resistor and a 12-pF capacitor, and $f = 2$ kHz. Find: (a) the series components that make up $\mathbf{Z}_x$, and (b) the parallel components that make up $\mathbf{Z}_x$.

Solution:

From Eq. (9.73),

$$\mathbf{Z}_x = \frac{\mathbf{Z}_3}{\mathbf{Z}_1}\mathbf{Z}_2 \tag{9.15.1}$$

where $\mathbf{Z}_x = R_x + jX_x$,

$$\mathbf{Z}_1 = 1000\ \Omega, \qquad \mathbf{Z}_2 = 4200\ \Omega \tag{9.15.2}$$

and

$$\mathbf{Z}_3 = R_3 \parallel \frac{1}{j\omega C_3} = \frac{\dfrac{R_3}{j\omega C_3}}{R_3 + 1/j\omega C_3} = \frac{R_3}{1 + j\omega R_3 C_3}$$

Since $R_3 = 1.5$ MΩ and $C_3 = 12$ pF,

$$\mathbf{Z}_3 = \frac{1.5 \times 10^6}{1 + j2\pi \times 2 \times 10^3 \times 1.5 \times 10^6 \times 12 \times 10^{-12}} = \frac{1.5 \times 10^6}{1 + j0.2262}$$

or

$$\mathbf{Z}_3 = 1.427 - j0.3228\ \text{M}\Omega \tag{9.15.3}$$

(a) Assuming that $\mathbf{Z}_x$ is made up of series components, we substitute Eqs. (9.15.2) and (9.15.3) in Eq. (9.15.1) and obtain

$$\begin{aligned} R_x + jX_x &= \frac{4200}{1000}(1.427 - j0.3228) \times 10^6 \\ &= (5.993 - j1.356)\ \text{M}\Omega \end{aligned}$$

Equating the real and imaginary parts yields $R_x = 5.993$ MΩ and a capacitive reactance

$$X_x = \frac{1}{\omega C} = 1.356 \times 10^6$$

or

$$C = \frac{1}{\omega X_x} = \frac{1}{2\pi \times 2 \times 10^3 \times 1.356 \times 10^6} = 58.69\ \text{pF}$$

(b) If $\mathbf{Z}_x$ is made up of parallel components, we notice that $\mathbf{Z}_3$ is also a parallel combination. Hence, Eq. (9.15.1) becomes

$$\mathbf{Z}_x = \frac{4200}{1000} R_3 \left\| \frac{1}{j\omega C_3} = 4.2R_3 \right\| \frac{1}{j\omega C_3} = 4.2\mathbf{Z}_3 \tag{9.15.4}$$

This simply means that the unknown impedance $\mathbf{Z}_x$ is 4.2 times $\mathbf{Z}_3$. Since $\mathbf{Z}_3$ consists of R_3 and $X_3 = 1/\omega C_3$, there are many ways we can get $4.2\mathbf{Z}_3$. Therefore, there is no unique answer to the problem. If we suppose that $4.2 = 3 \times 1.4$ and we decide to multiply R_3 by 1.4 while multiplying X_3 by 3, then the answer is

$$R_x = 1.4R_3 = 2.1\ \text{M}\Omega$$

and

$$X_x = \frac{1}{\omega C_x} = 3X_3 = \frac{3}{\omega C_3} \qquad \Longrightarrow \qquad C_x = \frac{1}{3}C_3 = 4\ \text{pF}$$

Alternatively, we may decide to multiply R_3 by 3 while multiplying X_x by 1.4 and obtain $R_x = 4.5$ MΩ and $C_x = C_3/1.4 = 8.571$ pF. Of course, there are several other possibilities. In a situation like this when there is no unique solution, care must be taken to select reasonably sized component values whenever possible.

PRACTICE PROBLEM 9.15

In the ac bridge circuit of Fig. 9.37, suppose that balance is achieved when $\mathbf{Z}_1$ is a 4.8-kΩ resistor, $\mathbf{Z}_2$ is a 10-Ω resistor in series with a 0.25-μH inductor, $\mathbf{Z}_3$ is a 12-kΩ resistor, and $f = 6$ MHz. Determine the series components that make up $\mathbf{Z}_x$.

Answer: A 25-Ω resistor in series with a 0.625-μH inductor.

9.9 SUMMARY

1. A sinusoid is a signal in the form of the sine or cosine function. It has the general form

$$v(t) = V_m \cos(\omega t + \phi)$$

where V_m is the amplitude, $\omega = 2\pi f$ is the angular frequency, $(\omega t + \phi)$ is the argument, and ϕ is the phase.

2. A phasor is a complex quantity that represents both the magnitude and the phase of a sinusoid. Given the sinusoid $v(t) = V_m \cos(\omega t + \phi)$, its phasor $\mathbf{V}$ is

$$\mathbf{V} = V_m \angle \phi$$

3. In ac circuits, voltage and current phasors always have a fixed relation to one another at any moment of time. If $v(t) = V_m \cos(\omega t + \phi_v)$ represents the voltage through an element and $i(t) = I_m \cos(\omega t + \phi_i)$ represents the current through the element, then $\phi_i = \phi_v$ if the element is a resistor, ϕ_i leads ϕ_v by 90° if the element is a capacitor, and ϕ_i lags ϕ_v by 90° if the element is an inductor.

4. The impedance $\mathbf{Z}$ of a circuit is the ratio of the phasor voltage across it to the phasor current through it:

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = R(\omega) + jX(\omega)$$

The admittance $\mathbf{Y}$ is the reciprocal of impedance:

$$\mathbf{Z} = \frac{1}{\mathbf{Y}} = G(\omega) + jB(\omega)$$

Impedances are combined in series or in parallel the same way as resistances in series or parallel; that is, impedances in series add while admittances in parallel add.

5. For a resistor $\mathbf{Z} = R$, for an inductor $\mathbf{Z} = jX = j\omega L$, and for a capacitor $\mathbf{Z} = -jX = 1/j\omega C$.

6. Basic circuit laws (Ohm's and Kirchhoff's) apply to ac circuits in the same manner as they do for dc circuits; that is,

$$\mathbf{V} = \mathbf{ZI}$$
$$\sum \mathbf{I}_k = 0 \quad \text{(KCL)}$$
$$\sum \mathbf{V}_k = 0 \quad \text{(KVL)}$$

7. The techniques of voltage/current division, series/parallel combination of impedance/admittance, circuit reduction, and Y-Δ transformation all apply to ac circuit analysis.
8. AC circuits are applied in phase-shifters and bridges.

REVIEW QUESTIONS

9.1 Which of the following is *not* a right way to express the sinusoid $A\cos\omega t$?
(a) $A\cos 2\pi f t$ (b) $A\cos(2\pi t/T)$
(c) $A\cos\omega(t-T)$ (d) $A\sin(\omega t - 90°)$

9.2 A function that repeats itself after fixed intervals is said to be:
(a) a phasor (b) harmonic
(c) periodic (d) reactive

9.3 Which of these frequencies has the shorter period?
(a) 1 krad/s (b) 1 kHz

9.4 If $v_1 = 30\sin(\omega t + 10°)$ and $v_2 = 20\sin(\omega t + 50°)$, which of these statements are true?
(a) v_1 leads v_2 (b) v_2 leads v_1
(c) v_2 lags v_1 (d) v_1 lags v_2
(e) v_1 and v_2 are in phase

9.5 The voltage across an inductor leads the current through it by 90°.
(a) True (b) False

9.6 The imaginary part of impedance is called:
(a) resistance (b) admittance
(c) susceptance (d) conductance
(e) reactance

9.7 The impedance of a capacitor increases with increasing frequency.
(a) True (b) False

9.8 At what frequency will the output voltage $v_o(t)$ in Fig. 9.39 be equal to the input voltage $v(t)$?
(a) 0 rad/s (b) 1 rad/s (c) 4 rad/s
(d) ∞ rad/s (e) none of the above

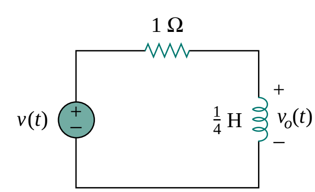

Figure 9.39 For Review Question 9.8.

9.9 A series RC circuit has $V_R = 12$ V and $V_C = 5$ V. The supply voltage is:
(a) −7 V (b) 7 V (c) 13 V (d) 17 V

9.10 A series RCL circuit has $R = 30\ \Omega$, $X_C = -50\ \Omega$, and $X_L = 90\ \Omega$. The impedance of the circuit is:
(a) $30 + j140\ \Omega$ (b) $30 + j40\ \Omega$
(c) $30 - j40\ \Omega$ (d) $-30 - j40\ \Omega$
(e) $-30 + j40\ \Omega$

Answers: 9.1d, 9.2c, 9.3b, 9.4b,d, 9.5a, 9.6e, 9.7b, 9.8d, 9.9c, 9.10b.

PROBLEMS

Section 9.2 Sinusoids

9.1 In a linear circuit, the voltage source is

$$v_s = 12\sin(10^3 t + 24°)\ \text{V}$$

(a) What is the angular frequency of the voltage?
(b) What is the frequency of the source?
(c) Find the period of the voltage.
(d) Express v_s in cosine form.
(e) Determine v_s at $t = 2.5$ ms.

9.2 A current source in a linear circuit has

$$i_s = 8\cos(500\pi t - 25°)\ \text{A}$$

(a) What is the amplitude of the current?
(b) What is the angular frequency?

(c) Find the frequency of the current.
(d) Calculate i_s at $t = 2$ ms.

9.3 Express the following functions in cosine form:
(a) $4\sin(\omega t - 30^\circ)$ (b) $-2\sin 6t$
(c) $-10\sin(\omega t + 20^\circ)$

9.4 (a) Express $v = 8\cos(7t + 15^\circ)$ in sine form.
(b) Convert $i = -10\sin(3t - 85^\circ)$ to cosine form.

9.5 Given $v_1 = 20\sin(\omega t + 60^\circ)$ and $v_2 = 60\cos(\omega t - 10^\circ)$, determine the phase angle between the two sinusoids and which one lags the other.

9.6 For the following pairs of sinusoids, determine which one leads and by how much.
(a) $v(t) = 10\cos(4t - 60^\circ)$ and $i(t) = 4\sin(4t + 50^\circ)$
(b) $v_1(t) = 4\cos(377t + 10^\circ)$ and $v_2(t) = -20\cos 377t$
(c) $x(t) = 13\cos 2t + 5\sin 2t$ and $y(t) = 15\cos(2t - 11.8^\circ)$

Section 9.3 Phasors

9.7 If $f(\phi) = \cos\phi + j\sin\phi$, show that $f(\phi) = e^{j\phi}$.

9.8 Calculate these complex numbers and express your results in rectangular form:

(a) $\dfrac{15\angle 45^\circ}{3 - j4} + j2$

(b) $\dfrac{8\angle -20^\circ}{(2 + j)(3 - j4)} + \dfrac{10}{-5 + j12}$

(c) $10 + (8\angle 50^\circ)(5 - j12)$

9.9 Evaluate the following complex numbers and express your results in rectangular form:

(a) $2 + \dfrac{3 + j4}{5 - j8}$ (b) $4\angle -10^\circ + \dfrac{1 - j2}{3\angle 6^\circ}$

(c) $\dfrac{8\angle 10^\circ + 6\angle -20^\circ}{9\angle 80^\circ - 4\angle 50^\circ}$

9.10 Given the complex numbers $z_1 = -3 + j4$ and $z_2 = 12 + j5$, find:
(a) $z_1 z_2$ (b) $\dfrac{z_1}{z_2^*}$ (c) $\dfrac{z_1 + z_2}{z_1 - z_2}$

9.11 Let $\mathbf{X} = 8\angle 40^\circ$ and $\mathbf{Y} = 10\angle -30^\circ$. Evaluate the following quantities and express your results in polar form.
(a) $(\mathbf{X} + \mathbf{Y})\mathbf{X}^*$ (b) $(\mathbf{X} - \mathbf{Y})^*$ (c) $(\mathbf{X} + \mathbf{Y})/\mathbf{X}$

9.12 Evaluate these determinants:

(a) $\begin{vmatrix} 10 + j6 & 2 - j3 \\ -5 & -1 + j \end{vmatrix}$

(b) $\begin{vmatrix} 20\angle -30^\circ & -4\angle -10^\circ \\ 16\angle 0^\circ & 3\angle 45^\circ \end{vmatrix}$

(c) $\begin{vmatrix} 1 - j & -j & 0 \\ j & 1 & -j \\ 1 & j & 1 + j \end{vmatrix}$

9.13 Transform the following sinusoids to phasors:
(a) $-10\cos(4t + 75^\circ)$ (b) $5\sin(20t - 10^\circ)$
(c) $4\cos 2t + 3\sin 2t$

9.14 Express the sum of the following sinusoidal signals in the form of $A\cos(\omega t + \theta)$ with $A > 0$ and $0 < \theta < 360^\circ$.
(a) $8\cos(5t - 30^\circ) + 6\cos 5t$
(b) $20\cos(120\pi t + 45^\circ) - 30\sin(120\pi t + 20^\circ)$
(c) $4\sin 8t + 3\sin(8t - 10^\circ)$

9.15 Obtain the sinusoids corresponding to each of the following phasors:
(a) $\mathbf{V}_1 = 60\angle 15^\circ$, $\omega = 1$
(b) $\mathbf{V}_2 = 6 + j8$, $\omega = 40$
(c) $\mathbf{I}_1 = 2.8e^{-j\pi/3}$, $\omega = 377$
(d) $\mathbf{I}_2 = -0.5 - j1.2$, $\omega = 10^3$

9.16 Using phasors, find:
(a) $3\cos(20t + 10^\circ) - 5\cos(20t - 30^\circ)$
(b) $40\sin 50t + 30\cos(50t - 45^\circ)$
(c) $20\sin 400t + 10\cos(400t + 60^\circ) - 5\sin(400t - 20^\circ)$

9.17 Find a single sinusoid corresponding to each of these phasors:
(a) $\mathbf{V} = 40\angle -60^\circ$
(b) $\mathbf{V} = -30\angle 10^\circ + 50\angle 60^\circ$
(c) $\mathbf{I} = j6e^{-j10^\circ}$ (d) $\mathbf{I} = \dfrac{2}{j} + 10\angle -45^\circ$

9.18 Find $v(t)$ in the following integrodifferential equations using the phasor approach:
(a) $v(t) + \int v\,dt = 10\cos t$
(b) $\dfrac{dv}{dt} + 5v(t) + 4\int v\,dt = 20\sin(4t + 10^\circ)$

9.19 Using phasors, determine $i(t)$ in the following equations:
(a) $2\dfrac{di}{dt} + 3i(t) = 4\cos(2t - 45^\circ)$
(b) $10\int i\,dt + \dfrac{di}{dt} + 6i(t) = 5\cos(5t + 22^\circ)$

9.20 The loop equation for a series RLC circuit gives

$$\frac{di}{dt} + 2i + \int_{-\infty}^{t} i\,dt = \cos 2t$$

Assuming that the value of the integral at $t = -\infty$ is zero, find $i(t)$ using the phasor method.

9.21 A parallel RLC circuit has the node equation

$$\frac{dv}{dt} + 50v + 100\int v\,dt = 110\cos(377t - 10^\circ)$$

Determine $v(t)$ using the phasor method. You may assume that the value of the integral at $t = -\infty$ is zero.

Section 9.4 Phasor Relationships for Circuit Elements

9.22 Determine the current that flows through an 8-Ω resistor connected to a voltage source $v_s = 110\cos 377t$ V.

9.23 What is the instantaneous voltage across a 2-μF capacitor when the current through it is $i = 4\sin(10^6 t + 25^\circ)$ A?

9.24 The voltage across a 4-mH inductor is $v = 60\cos(500t - 65^\circ)$ V. Find the instantaneous current through it.

9.25 A current source of $i(t) = 10\sin(377t + 30^\circ)$ A is applied to a single-element load. The resulting voltage across the element is $v(t) = -65\cos(377t + 120^\circ)$ V. What type of element is this? Calculate its value.

9.26 Two elements are connected in series as shown in Fig. 9.40. If $i = 12\cos(2t - 30^\circ)$ A, find the element values.

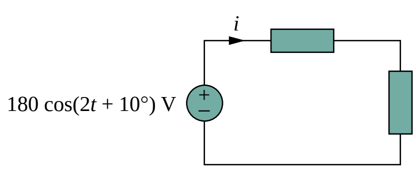

Figure 9.40 For Prob. 9.26.

9.27 A series RL circuit is connected to a 110-V ac source. If the voltage across the resistor is 85 V, find the voltage across the inductor.

9.28 What value of ω will cause the forced response v_o in Fig. 9.41 to be zero?

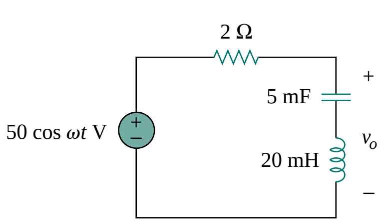

Figure 9.41 For Prob. 9.28.

Section 9.5 Impedance and Admittance

9.29 If $v_s = 5\cos 2t$ V in the circuit of Fig. 9.42, find v_o.

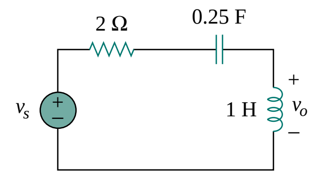

Figure 9.42 For Prob. 9.29.

9.30 Find i_x when $i_s = 2\sin 5t$ A is supplied to the circuit in Fig. 9.43.

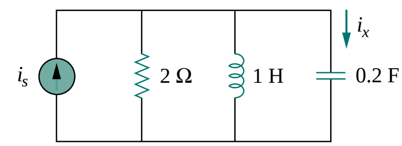

Figure 9.43 For Prob. 9.30.

9.31 Find $i(t)$ and $v(t)$ in each of the circuits of Fig. 9.44.

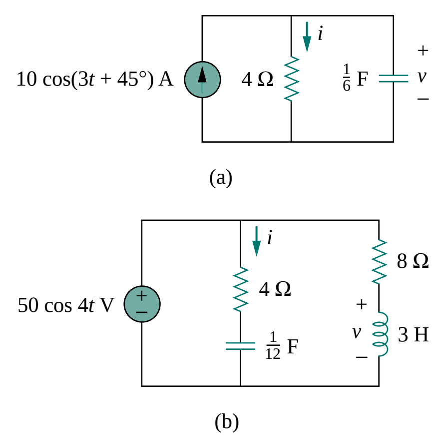

Figure 9.44 For Prob. 9.31.

9.32 Calculate $i_1(t)$ and $i_2(t)$ in the circuit of Fig. 9.45 if the source frequency is 60 Hz.

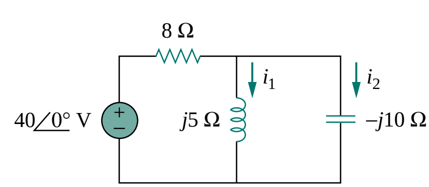

Figure 9.45 For Prob. 9.32.

9.33 In the circuit of Fig. 9.46, find i_o when:
(a) $\omega = 1$ rad/s (b) $\omega = 5$ rad/s
(c) $\omega = 10$ rad/s

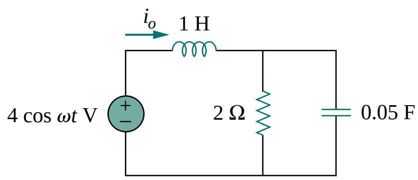

Figure 9.46 For Prob. 9.33.

9.34 Find $v(t)$ in the RLC circuit of Fig. 9.47.

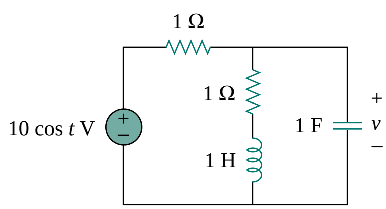

Figure 9.47 For Prob. 9.34.

9.35 Calculate $v_o(t)$ in the circuit in Fig. 9.48.

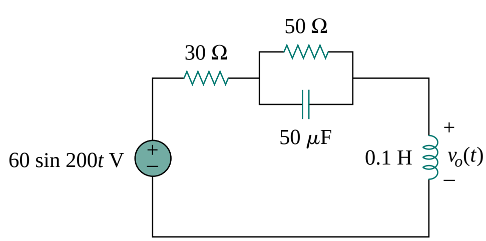

Figure 9.48 For Prob. 9.35.

9.36 Determine $i_o(t)$ in the RLC circuit of Fig. 9.49.

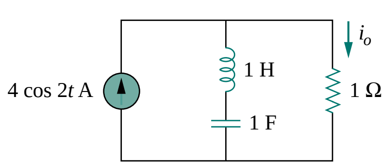

Figure 9.49 For Prob. 9.36.

9.37 Calculate $i(t)$ in the circuit of Fig. 9.50.

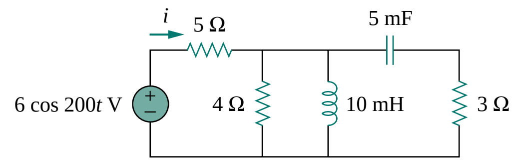

Figure 9.50 For Prob. 9.37.

9.38 Find current $\mathbf{I}_o$ in the network of Fig. 9.51.

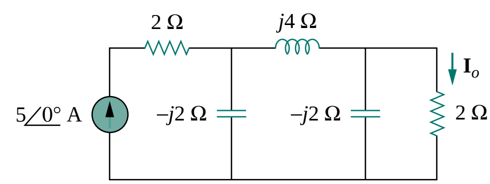

Figure 9.51 For Prob. 9.38.

9.39 If $i_s = 5\cos(10t + 40^\circ)$ A in the circuit in Fig. 9.52, find i_o.

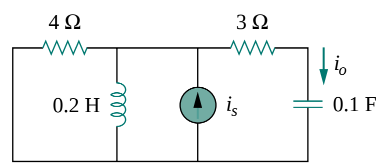

Figure 9.52 For Prob. 9.39.

9.40 Find $v_s(t)$ in the circuit of Fig. 9.53 if the current i_x through the 1-Ω resistor is $0.5 \sin 200t$ A.

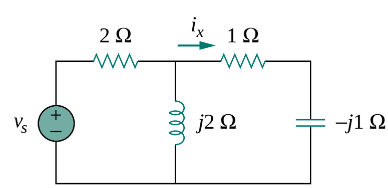

Figure 9.53 For Prob. 9.40.

9.41 If the voltage v_o across the 2-Ω resistor in the circuit of Fig. 9.54 is $10 \cos 2t$ V, obtain i_s.

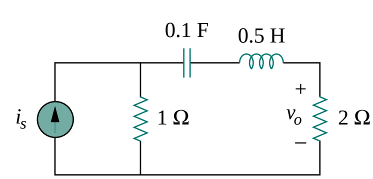

Figure 9.54 For Prob. 9.41.

9.42 If $\mathbf{V}_o = 8\angle 30^\circ$ V in the circuit of Fig. 9.55, find $\mathbf{I}_s$.

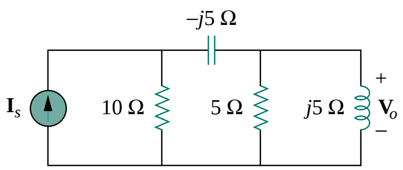

Figure 9.55 For Prob. 9.42.

9.43 In the circuit of Fig. 9.56, find $\mathbf{V}_s$ if $\mathbf{I}_o = 2\angle 0^\circ$ A.

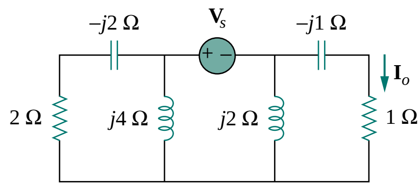

Figure 9.56 For Prob. 9.43.

9.44 Find $\mathbf{Z}$ in the network of Fig. 9.57, given that $\mathbf{V}_o = 4\angle 0^\circ$ V.

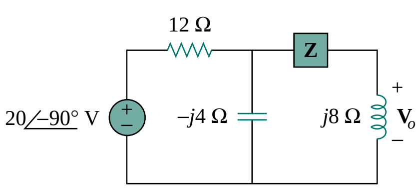

Figure 9.57 For Prob. 9.44.

Section 9.7 Impedance Combinations

9.45 At $\omega = 50$ rad/s, determine $\mathbf{Z}_{\text{in}}$ for each of the circuits in Fig. 9.58.

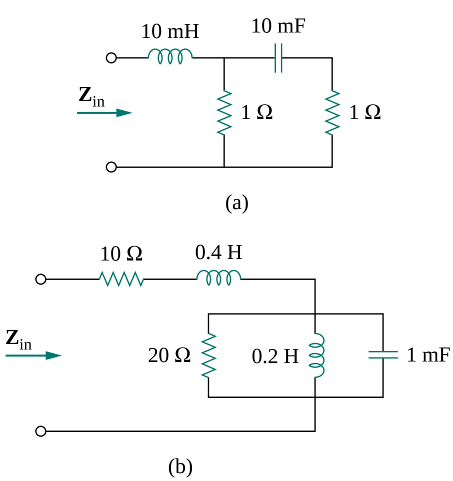

Figure 9.58 For Prob. 9.45.

9.46 Calculate $\mathbf{Z}_{\text{eq}}$ for the circuit in Fig. 9.59.

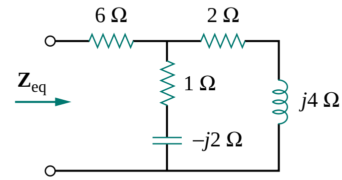

Figure 9.59 For Prob. 9.46.

9.47 Find $\mathbf{Z}_{\text{eq}}$ in the circuit of Fig. 9.60.

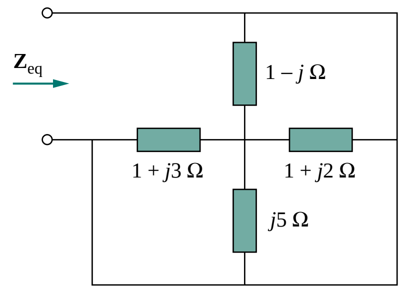

Figure 9.60 For Prob. 9.47.

9.48 For the circuit in Fig. 9.61, find the input impedance $\mathbf{Z}_{\text{in}}$ at 10 krad/s.

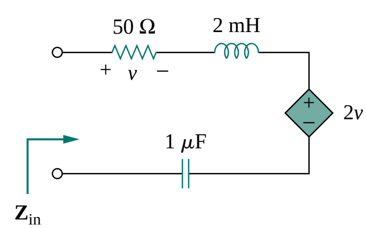

Figure 9.61 For Prob. 9.48.

9.49 Determine $\mathbf{I}$ and $\mathbf{Z}_T$ for the circuit in Fig. 9.62.

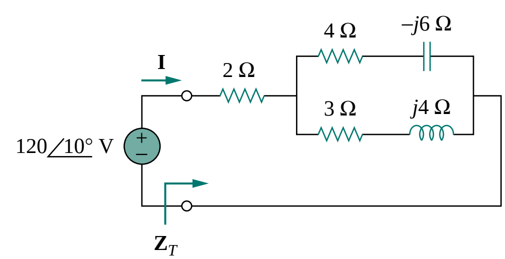

Figure 9.62 For Prob. 9.49.

9.50 For the circuit in Fig. 9.63, calculate $\mathbf{Z}_T$ and $\mathbf{V}_{ab}$.

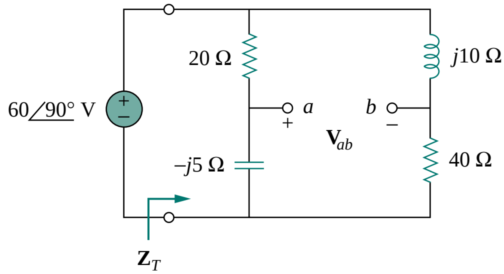

Figure 9.63 For Prob. 9.50.

9.51 At $\omega = 10^3$ rad/s, find the input admittance of each of the circuits in Fig. 9.64.

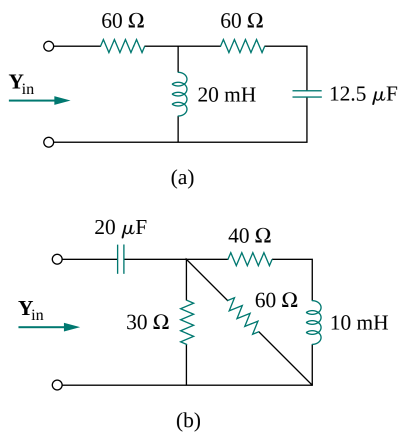

Figure 9.64 For Prob. 9.51.

9.52 Determine $\mathbf{Y}_{eq}$ for the circuit in Fig. 9.65.

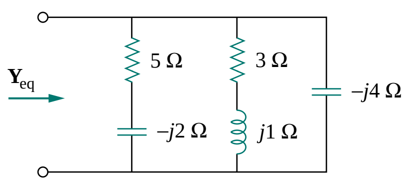

Figure 9.65 For Prob. 9.52.

9.53 Find the equivalent admittance $\mathbf{Y}_{eq}$ of the circuit in Fig. 9.66.

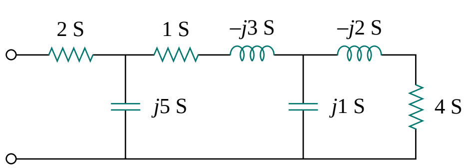

Figure 9.66 For Prob. 9.53.

9.54 Find the equivalent impedance of the circuit in Fig. 9.67.

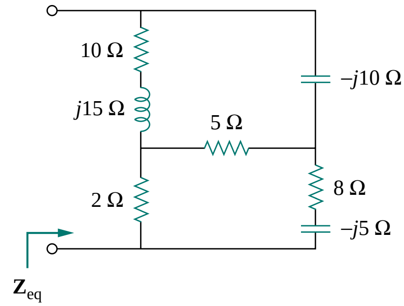

Figure 9.67 For Prob. 9.54.

9.55 Obtain the equivalent impedance of the circuit in Fig. 9.68.

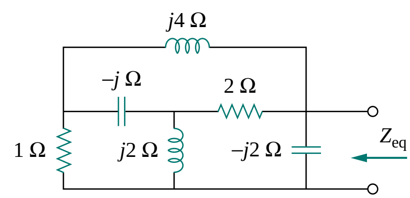

Figure 9.68 For Prob. 9.55.

9.56 Calculate the value of $\mathbf{Z}_{ab}$ in the network of Fig. 9.69.

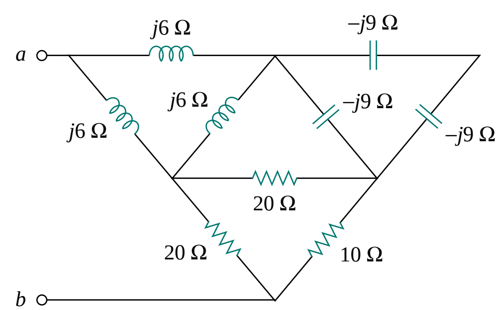

Figure 9.69 For Prob. 9.56.

9.57 Determine the equivalent impedance of the circuit in Fig. 9.70.

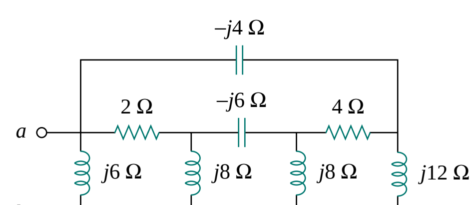

Figure 9.70 For Prob. 9.57.

Section 9.8 Applications

9.58 Design an RL circuit to provide a 90° leading phase shift.

9.59 Design a circuit that will transform a sinusoidal input to a cosinusoidal output.

9.60 Refer to the RC circuit in Fig. 9.71.
(a) Calculate the phase shift at 2 MHz.
(b) Find the frequency where the phase shift is 45°.

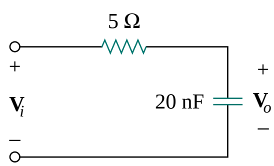

Figure 9.71 For Prob. 9.60.

9.61 (a) Calculate the phase shift of the circuit in Fig. 9.72.
(b) State whether the phase shift is leading or lagging (output with respect to input).
(c) Determine the magnitude of the output when the input is 120 V.

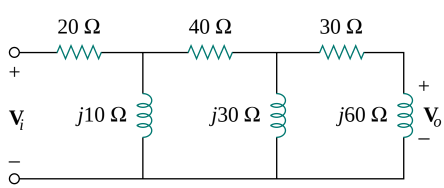

Figure 9.72 For Prob. 9.61.

9.62 Consider the phase-shifting circuit in Fig. 9.73. Let $\mathbf{V}_i = 120$ V operating at 60 Hz. Find:
(a) $\mathbf{V}_o$ when R is maximum
(b) $\mathbf{V}_o$ when R is minimum
(c) the value of R that will produce a phase shift of 45°

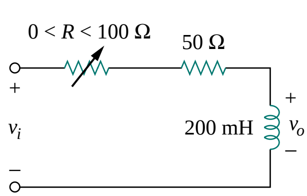

Figure 9.73 For Prob. 9.62.

9.63 The ac bridge in Fig. 9.37 is balanced when $R_1 = 400\ \Omega$, $R_2 = 600\ \Omega$, $R_3 = 1.2\ \text{k}\Omega$, and $C_2 = 0.3\ \mu\text{F}$. Find R_x and C_x.

9.64 A capacitance bridge balances when $R_1 = 100\ \Omega$, $R_2 = 2\ \text{k}\Omega$, and $C_s = 40\ \mu\text{F}$. What is C_x, the capacitance of the capacitor under test?

9.65 An inductive bridge balances when $R_1 = 1.2\ \text{k}\Omega$, $R_2 = 500\ \Omega$, and $L_s = 250$ mH. What is the value of L_x, the inductance of the inductor under test?

9.66 The ac bridge shown in Fig. 9.74 is known as a *Maxwell bridge* and is used for accurate measurement of inductance and resistance of a coil in terms of a standard capacitance C_s. Show that when the bridge is balanced,

$$L_x = R_2 R_3 C_s \qquad \text{and} \qquad R_x = \frac{R_2}{R_1} R_3$$

Find L_x and R_x for $R_1 = 40\ \text{k}\Omega$, $R_2 = 1.6\ \text{k}\Omega$, $R_3 = 4\ \text{k}\Omega$, and $C_s = 0.45\ \mu\text{F}$.

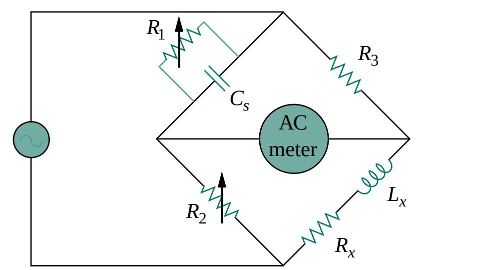

Figure 9.74 Maxwell bridge; for Prob. 9.66.

9.67 The ac bridge circuit of Fig. 9.75 is called a *Wien bridge*. It is used for measuring the frequency of a source. Show that when the bridge is balanced,

$$f = \frac{1}{2\pi\sqrt{R_2 R_4 C_2 C_4}}$$

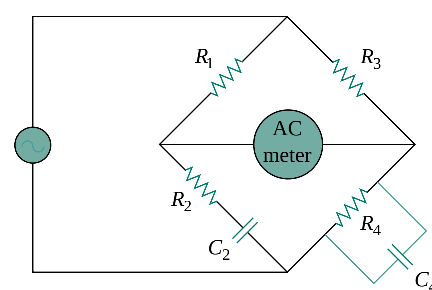

Figure 9.75 Wein bridge; for Prob. 9.67.

COMPREHENSIVE PROBLEMS

9.68 The circuit shown in Fig. 9.76 is used in a television receiver. What is the total impedance of this circuit?

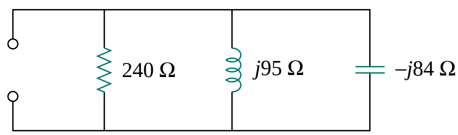

Figure 9.76 For Prob. 9.68.

9.69 The network in Fig. 9.77 is part of the schematic describing an industrial electronic sensing device. What is the total impedance of the circuit at 2 kHz?

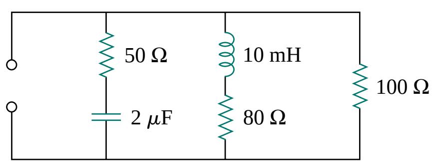

Figure 9.77 For Prob. 9.69.

9.70 A series audio circuit is shown in Fig. 9.78.

(a) What is the impedance of the circuit?

(b) If the frequency were halved, what would be the impedance of the circuit?

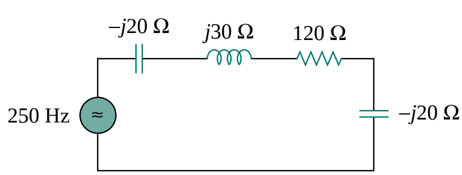

Figure 9.78 For Prob. 9.70.

9.71 An industrial load is modeled as a series combination of a capacitance and a resistance as shown in Fig. 9.79. Calculate the value of an inductance L across the series combination so that the net impedance is resistive at a frequency of 5 MHz.

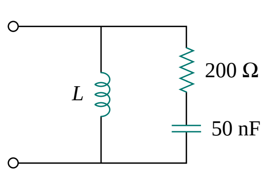

Figure 9.79 For Prob. 9.71.

9.72 An industrial coil is modeled as a series combination of an inductance L and resistance R, as shown in Fig. 9.80. Since an ac voltmeter measures only the magnitude of a sinusoid, the following measurements are taken at 60 Hz when the circuit operates in the steady state:

$$|\mathbf{V}_s| = 145 \text{ V}, \qquad |\mathbf{V}_1| = 50 \text{ V}, \qquad |\mathbf{V}_o| = 110 \text{ V}$$

Use these measurements to determine the values of L and R.

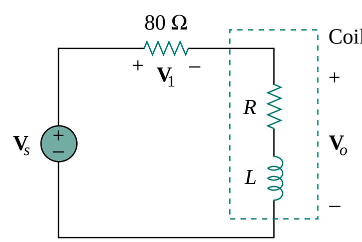

Figure 9.80 For Prob. 9.72.

9.73 Figure 9.81 shows a parallel combination of an inductance and a resistance. If it is desired to connect a capacitor in series with the parallel combination such that the net impedance is resistive at 10 MHz, what is the required value of C?

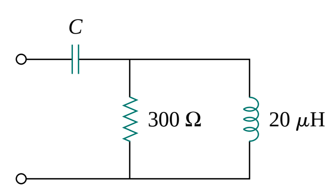

Figure 9.81 For Prob. 9.73.

9.74 A power transmission system is modeled as shown in Fig. 9.82. Given the source voltage $\mathbf{V}_s = 115\angle 0^\circ$ V, source impedance $\mathbf{Z}_s = 1 + j0.5\ \Omega$, line impedance $\mathbf{Z}_\ell = 0.4 + j0.3\ \Omega$, and load impedance $\mathbf{Z}_L = 23.2 + j18.9\ \Omega$, find the load current $\mathbf{I}_L$.

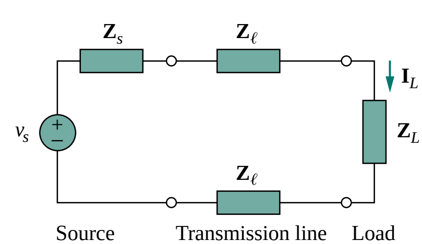

Figure 9.82 For Prob. 9.74.

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