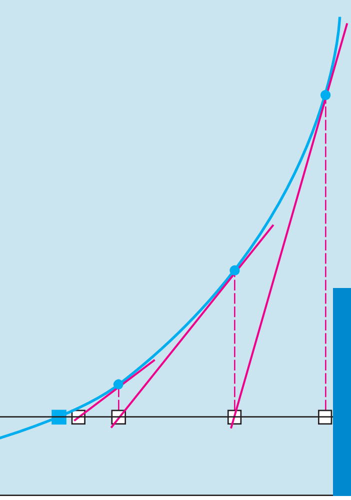


Chapter 4

Applications of Derivatives

OVERVIEW This chapter studies some of the important applications of derivatives. We learn how derivatives are used to find extreme values of functions, to determine and analyze the shapes of graphs, to calculate limits of fractions whose numerators and denominators both approach zero or infinity, and to find numerically where a function equals zero. We also consider the process of recovering a function from its derivative. The key to many of these accomplishments is the Mean Value Theorem, a theorem whose corollaries provide the gateway to integral calculus in Chapter 5.

4.1 Extreme Values of Functions

This section shows how to locate and identify extreme values of a continuous function from its derivative. Once we can do this, we can solve a variety of *optimization problems* in which we find the optimal (best) way to do something in a given situation.

DEFINITIONS Absolute Maximum, Absolute Minimum

Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on D at c if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in } D.$$

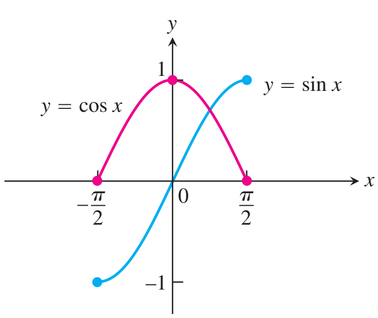


FIGURE 4.1 Absolute extrema for the sine and cosine functions on $[-\pi/2, \pi/2]$. These values can depend on the domain of a function.

Absolute maximum and minimum values are called absolute **extrema** (plural of the Latin *extremum*). Absolute extrema are also called **global** extrema, to distinguish them from *local extrema* defined below.

For example, on the closed interval $[-\pi/2, \pi/2]$ the function $f(x) = \cos x$ takes on an absolute maximum value of 1 (once) and an absolute minimum value of 0 (twice). On the same interval, the function $g(x) = \sin x$ takes on a maximum value of 1 and a minimum value of -1 (Figure 4.1).

Functions with the same defining rule can have different extrema, depending on the domain.

EXAMPLE 1 Exploring Absolute Extrema

The absolute extrema of the following functions on their domains can be seen in Figure 4.2.

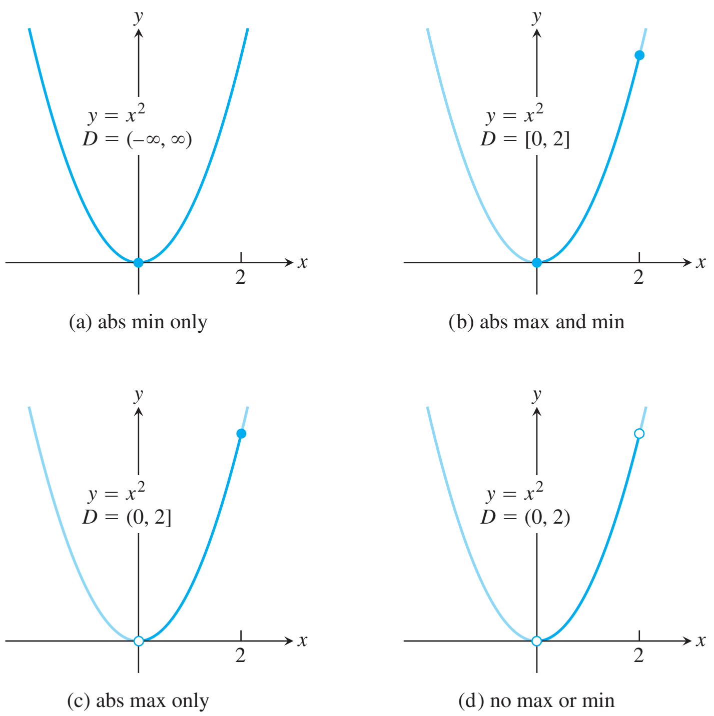


FIGURE 4.2 Graphs for Example 1.

Function rule	**Domain *D***	**Absolute extrema on *D***
(a) $y = x^2$	$(-\infty, \infty)$	No absolute maximum. Absolute minimum of 0 at $x = 0$.
(b) $y = x^2$	$[0, 2]$	Absolute maximum of 4 at $x = 2$. Absolute minimum of 0 at $x = 0$.
(c) $y = x^2$	$(0, 2]$	Absolute maximum of 4 at $x = 2$. No absolute minimum.
(d) $y = x^2$	$(0, 2)$	No absolute extrema. ■

HISTORICAL BIOGRAPHY

Daniel Bernoulli
(1700–1789)

The following theorem asserts that a function which is continuous at every point of a closed interval $[a, b]$ has an absolute maximum and an absolute minimum value on the interval. We always look for these values when we graph a function.

THEOREM 1 The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other x in $[a, b]$ (Figure 4.3).

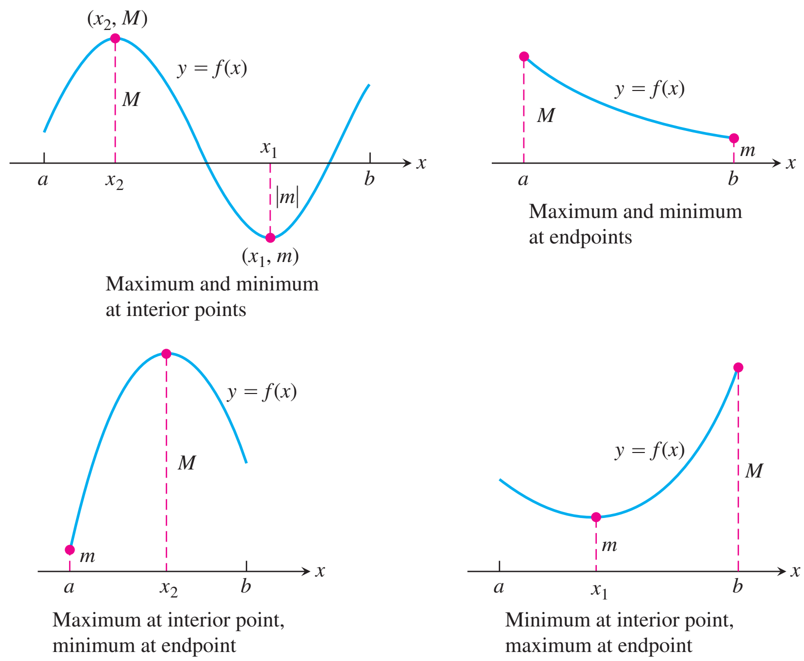


FIGURE 4.3 Some possibilities for a continuous function's maximum and minimum on a closed interval $[a, b]$.

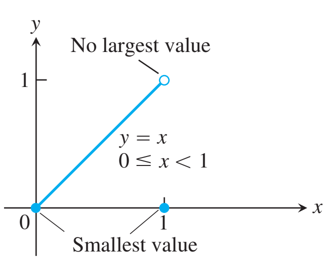


FIGURE 4.4 Even a single point of discontinuity can keep a function from having either a maximum or minimum value on a closed interval. The function

$$y = \begin{cases} x, & 0 \leq x < 1 \\ 0, & x = 1 \end{cases}$$

is continuous at every point of $[0, 1]$ except $x = 1$, yet its graph over $[0, 1]$ does not have a highest point.

The proof of The Extreme Value Theorem requires a detailed knowledge of the real number system (see Appendix 4) and we will not give it here. Figure 4.3 illustrates possible locations for the absolute extrema of a continuous function on a closed interval $[a, b]$. As we observed for the function $y = \cos x$, it is possible that an absolute minimum (or absolute maximum) may occur at two or more different points of the interval.

The requirements in Theorem 1 that the interval be closed and finite, and that the function be continuous, are key ingredients. Without them, the conclusion of the theorem need not hold. Example 1 shows that an absolute extreme value may not exist if the interval fails to be both closed and finite. Figure 4.4 shows that the continuity requirement cannot be omitted.

Local (Relative) Extreme Values

Figure 4.5 shows a graph with five points where a function has extreme values on its domain $[a, b]$. The function's absolute minimum occurs at a even though at e the function's value is

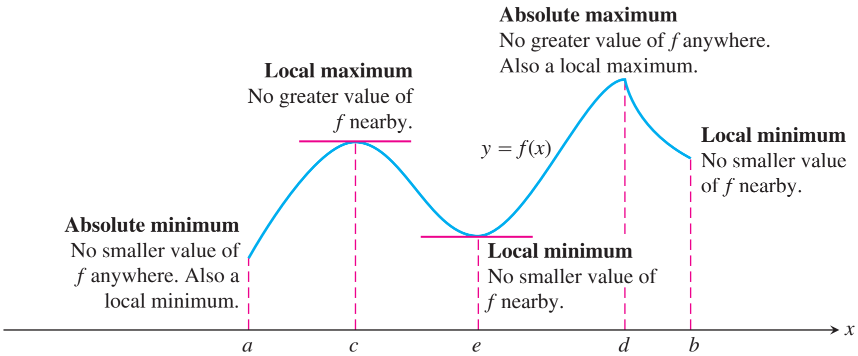


FIGURE 4.5 How to classify maxima and minima.

smaller than at any other point *nearby*. The curve rises to the left and falls to the right around c, making $f(c)$ a maximum locally. The function attains its absolute maximum at d.

DEFINITIONS Local Maximum, Local Minimum

A function f has a **local maximum** value at an interior point c of its domain if

$$f(x) \le f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$

A function f has a **local minimum** value at an interior point c of its domain if

$$f(x) \ge f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$

We can extend the definitions of local extrema to the endpoints of intervals by defining f to have a **local maximum** or **local minimum** value *at an endpoint* c if the appropriate inequality holds for all x in some half-open interval in its domain containing c. In Figure 4.5, the function f has local maxima at c and d and local minima at a, e, and b. Local extrema are also called **relative extrema**.

An absolute maximum is also a local maximum. Being the largest value overall, it is also the largest value in its immediate neighborhood. Hence, *a list of all local maxima will automatically include the absolute maximum if there is one*. Similarly, *a list of all local minima will include the absolute minimum if there is one*.

Finding Extrema

The next theorem explains why we usually need to investigate only a few values to find a function's extrema.

THEOREM 2 The First Derivative Theorem for Local Extreme Values

If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then

$$f'(c) = 0.$$

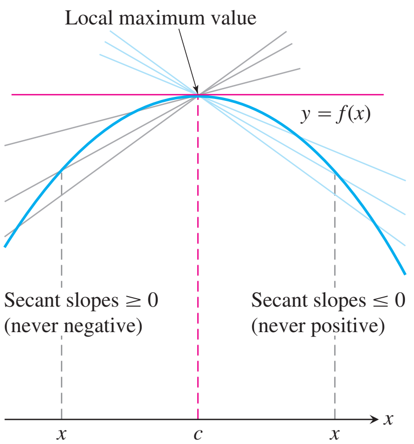


FIGURE 4.6 A curve with a local maximum value. The slope at c, simultaneously the limit of nonpositive numbers and nonnegative numbers, is zero.

Proof To prove that $f'(c)$ is zero at a local extremum, we show first that $f'(c)$ cannot be positive and second that $f'(c)$ cannot be negative. The only number that is neither positive nor negative is zero, so that is what $f'(c)$ must be.

To begin, suppose that f has a local maximum value at $x = c$ (Figure 4.6) so that $f(x) - f(c) \leq 0$ for all values of x near enough to c. Since c is an interior point of f's domain, $f'(c)$ is defined by the two-sided limit

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

This means that the right-hand and left-hand limits both exist at $x = c$ and equal $f'(c)$. When we examine these limits separately, we find that

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leq 0. \qquad \text{Because } (x - c) > 0 \text{ and } f(x) \leq f(c) \qquad (1)$$

Similarly,

$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0. \qquad \text{Because } (x - c) < 0 \text{ and } f(x) \leq f(c) \qquad (2)$$

Together, Equations (1) and (2) imply $f'(c) = 0$.

This proves the theorem for local maximum values. To prove it for local minimum values, we simply use $f(x) \geq f(c)$, which reverses the inequalities in Equations (1) and (2). ■

Theorem 2 says that a function's first derivative is always zero at an interior point where the function has a local extreme value and the derivative is defined. Hence the only places where a function f can possibly have an extreme value (local or global) are

1. interior points where $f' = 0$,
2. interior points where f' is undefined,
3. endpoints of the domain of f.

The following definition helps us to summarize.

DEFINITION Critical Point

An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f.

Thus the only domain points where a function can assume extreme values are critical points and endpoints.

Be careful not to misinterpret Theorem 2 because its converse is false. A differentiable function may have a critical point at $x = c$ without having a local extreme value there. For instance, the function $f(x) = x^3$ has a critical point at the origin and zero value there, but is positive to the right of the origin and negative to the left. So it cannot have a local extreme value at the origin. Instead, it has a *point of inflection* there. This idea is defined and discussed further in Section 4.4.

Most quests for extreme values call for finding the absolute extrema of a continuous function on a closed and finite interval. Theorem 1 assures us that such values exist; Theorem 2 tells us that they are taken on only at critical points and endpoints. Often we can simply list these points and calculate the corresponding function values to find what the largest and smallest values are, and where they are located.

How to Find the Absolute Extrema of a Continuous Function f on a Finite Closed Interval

1. Evaluate f at all critical points and endpoints.
2. Take the largest and smallest of these values.

EXAMPLE 2 Finding Absolute Extrema

Find the absolute maximum and minimum values of $f(x) = x^2$ on $[-2, 1]$.

Solution The function is differentiable over its entire domain, so the only critical point is where $f'(x) = 2x = 0$, namely $x = 0$. We need to check the function's values at $x = 0$ and at the endpoints $x = -2$ and $x = 1$:

$$\begin{aligned} &\text{Critical point value:} && f(0) = 0 \\ &\text{Endpoint values:} && f(-2) = 4 \\ & && f(1) = 1 \end{aligned}$$

The function has an absolute maximum value of 4 at $x = -2$ and an absolute minimum value of 0 at $x = 0$. ■

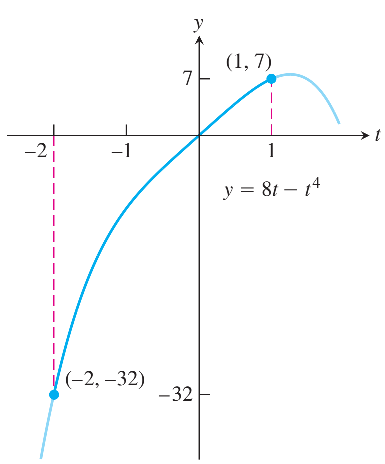


FIGURE 4.7 The extreme values of $g(t) = 8t - t^4$ on $[-2, 1]$ (Example 3).

EXAMPLE 3 Absolute Extrema at Endpoints

Find the absolute extrema values of $g(t) = 8t - t^4$ on $[-2, 1]$.

Solution The function is differentiable on its entire domain, so the only critical points occur where $g'(t) = 0$. Solving this equation gives

$$8 - 4t^3 = 0 \quad \text{or} \quad t = \sqrt[3]{2} > 1,$$

a point not in the given domain. The function's absolute extrema therefore occur at the endpoints, $g(-2) = -32$ (absolute minimum), and $g(1) = 7$ (absolute maximum). See Figure 4.7. ■

EXAMPLE 4 Finding Absolute Extrema on a Closed Interval

Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-2, 3]$.

Solution We evaluate the function at the critical points and endpoints and take the largest and smallest of the resulting values.

The first derivative

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

has no zeros but is undefined at the interior point $x = 0$. The values of f at this one critical point and at the endpoints are

$$\begin{aligned} &\text{Critical point value:} && f(0) = 0 \\ &\text{Endpoint values:} && f(-2) = (-2)^{2/3} = \sqrt[3]{4} \\ & && f(3) = (3)^{2/3} = \sqrt[3]{9}. \end{aligned}$$

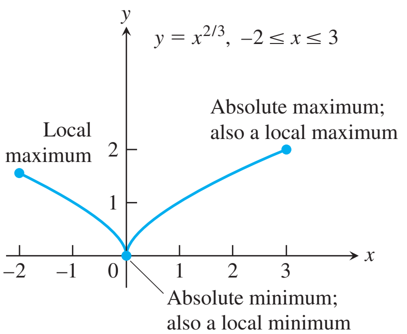


FIGURE 4.8 The extreme values of $f(x) = x^{2/3}$ on $[-2, 3]$ occur at $x = 0$ and $x = 3$ (Example 4).

We can see from this list that the function's absolute maximum value is $\sqrt[3]{9} \approx 2.08$, and it occurs at the right endpoint $x = 3$. The absolute minimum value is 0, and it occurs at the interior point $x = 0$. (Figure 4.8). ■

While a function's extrema can occur only at critical points and endpoints, not every critical point or endpoint signals the presence of an extreme value. Figure 4.9 illustrates this for interior points.

We complete this section with an example illustrating how the concepts we studied are used to solve a real-world optimization problem.

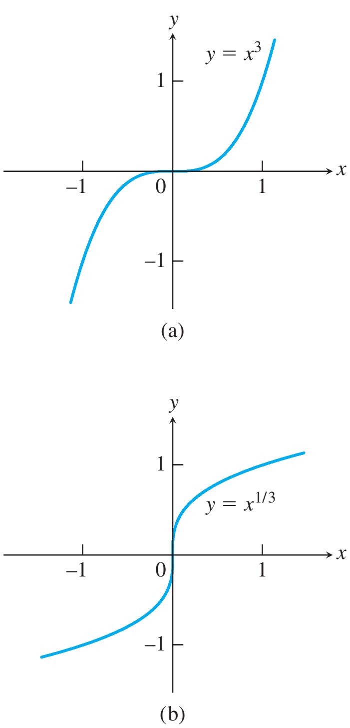


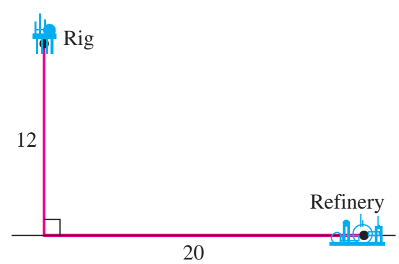
FIGURE 4.9 Critical points without extreme values. (a) $y' = 3x^2$ is 0 at $x = 0$, but $y = x^3$ has no extremum there. (b) $y' = (1/3)x^{-2/3}$ is undefined at $x = 0$, but $y = x^{1/3}$ has no extremum there.

EXAMPLE 5 Piping Oil from a Drilling Rig to a Refinery

A drilling rig 12 mi offshore is to be connected by pipe to a refinery onshore, 20 mi straight down the coast from the rig. If underwater pipe costs \$500,000 per mile and land-based pipe costs \$300,000 per mile, what combination of the two will give the least expensive connection?

Solution We try a few possibilities to get a feel for the problem:

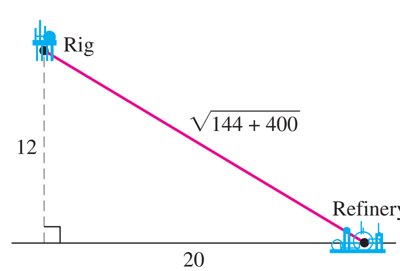
(a) *Smallest amount of underwater pipe*



Underwater pipe is more expensive, so we use as little as we can. We run straight to shore (12 mi) and use land pipe for 20 mi to the refinery.

$$\begin{aligned}\text{Dollar cost} &= 12(500{,}000) + 20(300{,}000)\\ &= 12{,}000{,}000\end{aligned}$$

(b) *All pipe underwater (most direct route)*

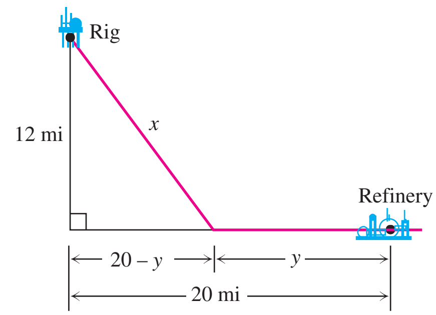


We go straight to the refinery underwater.

$$\begin{aligned}\text{Dollar cost} &= \sqrt{544}\,(500{,}000)\\ &\approx 11{,}661{,}900\end{aligned}$$

This is less expensive than plan (a).

(c) *Something in between*



Now we introduce the length x of underwater pipe and the length y of land-based pipe as variables. The right angle opposite the rig is the key to expressing the relationship between x and y, for the Pythagorean theorem gives

$$x^2 = 12^2 + (20 - y)^2$$
$$x = \sqrt{144 + (20 - y)^2}. \tag{3}$$

Only the positive root has meaning in this model.

The dollar cost of the pipeline is

$$c = 500{,}000x + 300{,}000y.$$

To express c as a function of a single variable, we can substitute for x, using Equation (3):

$$c(y) = 500{,}000\sqrt{144 + (20 - y)^2} + 300{,}000y.$$

Our goal now is to find the minimum value of $c(y)$ on the interval $0 \le y \le 20$. The first derivative of $c(y)$ with respect to y according to the Chain Rule is

$$c'(y) = 500{,}000 \cdot \frac{1}{2} \cdot \frac{2(20 - y)(-1)}{\sqrt{144 + (20 - y)^2}} + 300{,}000$$
$$= -500{,}000 \frac{20 - y}{\sqrt{144 + (20 - y)^2}} + 300{,}000.$$

Setting c' equal to zero gives

$$500{,}000\,(20 - y) = 300{,}000\sqrt{144 + (20 - y)^2}$$
$$\frac{5}{3}\left(20 - y\right) = \sqrt{144 + (20 - y)^2}$$
$$\frac{25}{9}\left(20 - y\right)^2 = 144 + (20 - y)^2$$
$$\frac{16}{9}\left(20 - y\right)^2 = 144$$
$$(20 - y) = \pm\frac{3}{4} \cdot 12 = \pm 9$$
$$y = 20 \pm 9$$
$$y = 11 \quad \text{or} \quad y = 29.$$

Only $y = 11$ lies in the interval of interest. The values of c at this one critical point and at the endpoints are

$$\begin{aligned} c(11) &= 10{,}800{,}000 \\ c(0) &= 11{,}661{,}900 \\ c(20) &= 12{,}000{,}000 \end{aligned}$$

The least expensive connection costs $10,800,000, and we achieve it by running the line underwater to the point on shore 11 mi from the refinery. ■