Fundamentals of COLLEGE GEOMETRY

SECOND EDITION

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Preface

Before revising *Fundamentals of College Geometry*, extensive questionnaires were sent to users of the earlier edition. A conscious effort has been made in this edition to incorporate the many fine suggestions given the respondents to the questionnaire. At the same time, I have attempted to preserve the features that made the earlier edition so popular.

The postulational structure of the text has been strengthened. Some definitions have been improved, making possible greater rigor in the development of the theorems. Particular stress has been continued in observing the distinction between equality and congruence. Symbols used for segments, intervals, rays, and half-lines have been changed in order that the symbols for the more common segment and ray will be easier to write. However, a symbol for the interval and half-line is introduced, which will still logically show their relations to the segment and ray.

Fundamental space concepts are introduced throughout the text in order to preserve continuity. However, the postulates and theorems on space geometry are kept to a minimum until Chapter 14. In this chapter, particular attention is given to mensuration problems dealing with geometric solids.

Greater emphasis has been placed on utilizing the principles of deductive logic covered in Chapter 2 in deriving geometric truths in subsequent chapters. Venn diagrams and truth tables have been expanded at a number of points throughout the text.

There is a wide variance throughout the United States in the time spent in geometry classes. Approximately two fifths of the classes meet three days a week. Another two fifths meet five days each week. The student who studies the first nine chapters of this text will have completed a well-rounded minimum course, including all of the fundamental concepts of plane and space geometry.

Each subsequent chapter in the book is written as a complete package, none of which is essential to the study of any of the other last five chapters, yet each will broaden the total background of the student. This will permit the instructor considerable latitude in adjusting his course to the time available and to the needs of his students.

Each chapter contains several sets of summary tests. These vary in type to include true-false tests, completion tests, problems tests, and proofs tests. A key for these tests and the problem sets throughout the text is available.

January 1969 *Edwin M. Hemmerling*

Preface to First Edition

During the past decade the entire approach to the teaching of geometry has been undergoing serious study by various nationally recognized professional groups. This book reflects many of their recommendations.

The style and objectives of this book are the same as those of my *College Plane Geometry*, out of which it has grown. Because I have added a significant amount of new material, however, and have increased the rigor employed, it has seemed desirable to give the book a new title. In *Fundamentals of College Geometry*, the presentation of the subject has been strengthened by the early introduction and continued use of the language and symbolism of sets as a unifying concept.

This book is designed for a semester's work. The student is introduced to the basic structure of geometry and is prepared to relate it to everyday experience as well as to subsequent study of mathematics.

The value of the precise use of language in stating definitions and hypotheses and in developing proofs is demonstrated. The student is helped to acquire an understanding of deductive thinking and a skill in applying it to mathematical situations. He is also given experience in the use of induction, analogy, and indirect methods of reasoning.

Abstract materials of geometry are related to experiences of daily life of the student. He learns to search for undefined terms and axioms in such areas of thinking as politics, sociology, and advertising. Examples of circular reasoning are studied.

In addition to providing for the promotion of proper attitudes, understandings, and appreciations, the book aids the student in learning to be critical in his listening, reading, and thinking. He is taught not to accept statements blindly but to think clearly before forming conclusions.

The chapter on coordinate geometry relates geometry and algebra. Properties of geometric figures are then determined analytically with the aid of algebra and the concept of one-to-one correspondence. A short chapter on trigonometry is given to relate ratio, similar polygons, and coordinate geometry.

Illustrative examples which aid in solving subsequent exercises are used liberally throughout the book. The student is able to learn a great deal of the material without the assistance of an instructor. Throughout the book he is afforded frequent opportunities for original and creative thinking. Many of the generous supply of exercises include developments which prepare for theorems that appear later in the text. The student is led to discover for himself proofs that follow.

The summary tests placed at the end of the book include completion, true-false, multiple-choice items, and problems. They afford the student and the instructor a ready means of measuring progress in the course.

Bakersfield, California, 1964 *Edwin M. Hemmerling*

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Basic Elements of Geometry

1.1. Historical background of geometry. Geometry is a study of the properties and measurements of figures composed of points and lines. It is a very old science and grew out of the needs of the people. The word geometry is derived from the Greek words *geo*, meaning "earth," and *metrein*, meaning "to measure." The early Egyptians and Babylonians (4000–3000 B.C.) were able to develop a collection of practical rules for measuring simple geometric figures and for determining their properties.

These rules were obtained inductively over a period of centuries of trial and error. They were not supported by any evidence of logical proof. Applications of these principles were found in the building of the Pyramids and the great Sphinx.

The irrigation systems devised by the early Egyptians indicate that they had an adequate knowledge of geometry as it may be applied in land surveying. The Babylonians were using geometric figures in tiles, walls, and decorations of their temples.

From Egypt and Babylonia the knowledge of geometry was taken to Greece. From the Greek people we have gained some of the greatest contributions to the advancement of mathematics. The Greek philosophers studied geometry not only for utilitarian benefits derived but for the esthetic and cultural advantages gained. The early Greeks thrived on a prosperous sea trade. This sea trade brought them not only wealth but also knowledge from other lands. These wealthy citizens of Greece had considerable time for fashionable debates and study on various topics of cultural interest because they had slaves to do most of their routine work. Usually theories and concepts brought back by returning seafarers from foreign lands made topics for lengthy and spirited debate by the Greeks.

Thus the Greeks became skilled in the art of logic and critical thinking. Among the more prominent Greeks contributing to this advancement were Thales of Miletus (640–546 B.C.), Pythagoras, a pupil of Thales (580?–500 B.C.), Plato (429–348 B.C.), Archimedes (287–212 B.C.), and Euclid (about 300 B.C.).

Euclid, who was a teacher of mathematics at the University of Alexandria, wrote the first comprehensive treatise on geometry. He entitled his text "Elements." Most of the principles now appearing in a modern text were present in Euclid's "Elements." His work has served as a model for most of the subsequent books written on geometry.

1.2. Why study geometry? The student beginning the study of this text may well ask, "What is geometry? What can I expect to gain from this study?"

Many leading institutions of higher learning have recognized that positive benefits can be gained by all who study this branch of mathematics. This is evident from the fact that they require study of geometry as a prerequisite to matriculation in those schools.

A study of geometry is an essential part of the training of the successful engineer, scientist, architect, and draftsman. The carpenter, machinist, tinsmith, stonecutter, artist, and designer all apply the facts of geometry in their trades. In this course the student will learn a great deal about geometric figures such as lines, angles, triangles, circles, and designs and patterns of many kinds.

One of the most important objectives derived from a study of geometry is making the student be more critical in his listening, reading, and thinking. In studying geometry he is led away from the practice of blind acceptance of statements and ideas and is taught to think clearly and critically before forming conclusions.

There are many other less direct benefits the student of geometry may gain. Among these one must include training in the exact use of the English language and in the ability to analyze a new situation or problem into its basic parts, and utilizing perseverence, originality, and logical reasoning in solving the problem. An appreciation for the orderliness and beauty of geometric forms that abound in man's works and of the creations of nature will be a by-product of the study of geometry. The student should also develop an awareness of the contributions of mathematics and mathematicians to our culture and civilization.

1.3. Sets and symbols. The idea of "set" is of great importance in mathematics. All of mathematics can be developed by starting with sets.

The word "set" is used to convey the idea of a collection of objects, usually with some common characteristic. These objects may be pieces of furniture

in a room, pupils enrolled in a geometry class, words in the English language, grains of sand on a beach, etc. These objects may also be distinguishable objects of our intuition or intellect, such as points, lines, numbers, and logical possibilities. The important feature of the set concept is that the collection of objects is to be regarded as a single entity. It is to be treated as a whole. Other words that convey the concept of set are "group," "bunch," "class," "aggregate," "covey," and "flock."

There are three ways of specifying a set. One is to give a rule by which it can be determined whether or not a given object is a *member* of the set; that is, the set is described. This method of specifying a set is called the *rule method*. The second method is to give a complete list of the members of the set. This is called the *roster method*. A third method frequently used for sets of real numbers is to graph the set on the number line. The members of a set are called its ***elements***. Thus "members" and "elements" can be used interchangeably.

It is customary to use braces { } to surround the elements of a set. For example, {1, 3, 5, 7} means the set whose members are the odd numbers 1, 3, 5, and 7. {Tom, Dick, Harry, Bill} might represent the members of a vocal quartet. A capital letter is often used to name or refer to a set. Thus, we could write $A = \{1, 3, 5, 7\}$ and $B =$ {Tom, Dick, Harry, Bill}.

A set may contain a finite number of elements, or an infinite number of elements. A finite set which contains no members is the *empty* or *null* set. The symbol for a null set is $\emptyset$ or { }. Thus, {even numbers ending in 5} = $\emptyset$. A set with a definite number* of members is a ***finite*** set. Thus, {5} is a finite set of which 5 is the only element. When the set contains many elements, it is customary to place inside the braces a description of the members of the set, e.g. {citizens of the United States}. A set with an infinite number of elements is termed an ***infinite set***. The natural numbers 1, 2, 3, ... form an infinite set. {0, 2, 4, 6, . . .} means the set of all nonnegative even numbers. It, too, is an infinite set.

In mathematics we use three dots (. . .) in two different ways in listing the elements of a set. For example

	Rule	**Roster**
1.	{integers greater than 10 and less than 100} Here the dots . . . mean "and so on up to and including."	{11, 12, 13, . . . , 99}
2.	{integers greater than 10} Here the dots . . . mean "and so on indefinitely."	{11, 12, 13, . . .}

*Zero is a definite number.

To symbolize the notion that 5 is an element of set A, we shall write $5 \in A$. If 6 is not a member of set A, we write $6 \notin A$, read "6 is not an element of set A."

Exercises

In exercises 1–12 it is given:

$A = \{1, 2, 3, 4, 5\}$. $B = \{6, 7, 8, 9, 10\}$.
$C = \{1, 2, 3, \ldots, 10\}$. $D = \{2, 4, 6, \ldots\}$.
$E = \emptyset$. $F = \{0\}$.
$G = \{5, 3, 2, 1, 4\}$. $H = \{1, 2, 3, \ldots\}$.

1. How many elements are in C? in E?
2. Give a rule describing H.
3. Do E and F contain the same elements?
4. Do A and G contain the same elements?
5. What elements are common to set A and set C?
6. What elements are common to set B and set D?
7. Which of the sets are finite?
8. Which of the sets are infinite?
9. What elements are common to A and B?
10. What elements are either in A or C or in both?
11. Insert in the following blank spaces the correct symbol $\in$ or $\notin$.
    (*a*) 3 ___ A (*b*) 3 ___ D (*c*) 0 ___ F
    (*d*) 0 ___ E (*e*) $\frac{3}{2}$ ___ H (*f*) 1002 ___ D
12. Give a rule describing F.

13–20. Use the roster method to describe each of the following sets.

Example. {whole numbers greater than 3 and less than 9}
Solution. {4, 5, 6, 7, 8}

13. {days of the week whose names begin with the letter T}
14. {even numbers between 29 and 39}
15. {whole numbers that are neither negative or positive}
16. {positive whole numbers}
17. {integers greater than 9}
18. {integers less than 1}
19. {months of the year beginning with the letter J}
20. {positive integers divisible by 3}

21–28. Use the rule method to describe each of the following sets.

Example. {California, Colorado, Connecticut}
Solution. {member states of the United States whose names begin with the letter C}

21. $\{a, e, i, o, u\}$
22. $\{a, b, c, \ldots, z\}$
23. {red, orange, yellow, green, blue, violet}
24. { }
25. {2, 4, 6, 8, 10}
26. {3, 4, 5, . . . , 50}
27. {−2, −4, −6, . . .}
28. {−6, −4, −2, 0, 2, 4, 6}

1.4. Relationships between sets. Two sets are *equal* if and only if they have the same elements. The equality between sets A and B is written $A = B$. The inequality of two sets is written $A \neq B$. For example, let set A be {whole numbers between $1\frac{1}{2}$ and $6\frac{3}{4}$} and let set B be {whole numbers between $1\frac{2}{3}$ and $6\frac{5}{6}$}. Then $A = B$ because the elements of both sets are the same: 2, 3, 4, 5, and 6. Here, then, is an example of two equal sets being described in two different ways. We could write {days of the week} or {Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday} as two ways of describing equal sets.

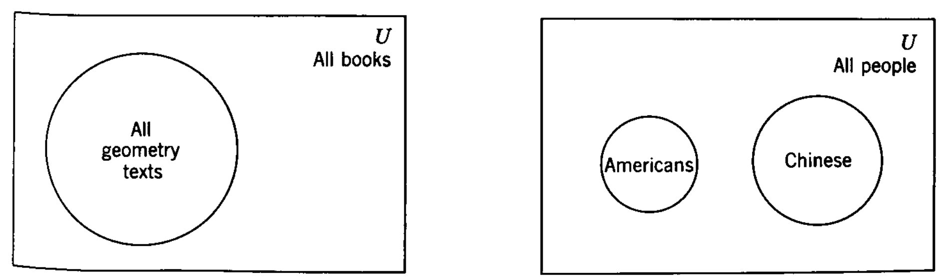
Often several sets are parts of a larger set. The set from which all other sets are drawn in a given discussion is called the *universal set*. The universal set, which may change from discussion to discussion, is often denoted by the letter U. In talking about the set of girls in a given geometry class, the universal set U might be all the students in the class, or it could be all the members of the student body of the given school, or all students in all schools, and so on.

Schematic representations to help illustrate properties of and operations with sets can be formed by drawing *Venn diagrams* (see Figs. 1.1*a* and 1.1*b*).

Here, points within a rectangle represent the elements of the universal set. Sets within the universal set are represented by points inside circles enclosed by the rectangle.

We shall frequently be interested in relationships between two or more sets. Consider the sets A and B where

$$A = \{2, 4, 6\} \quad \text{and} \quad B = \{1, 2, 3, 4, 5, 6\}.$$



(*a*) (*b*)

Fig. 1.1.

Definition: The set A is a *subset* of set B if, and only if, every element of set A is an element of set B. Thus, in the above illustration *A is a subset of B*. We write this relationship $A \subset B$ or $B \supset A$. In the illustration there are more elements in B than in A. This can be shown by the Venn diagram of Fig. 1.2. Notice, however, that our definition of subset does not stipulate it must contain fewer elements than does the given set. The subset can have exactly the same elements as the given set. In such a case, the two sets are equal and each is a subset of the other. Thus, any set is a subset of itself.

Illustrations

(*a*) Given $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Then $B \subset A$.
(*b*) Given $R = \{\text{integers}\}$ and $S = \{\text{odd integers}\}$. Then $S \subset R$.
(*c*) Given $C = \{\text{positive integers}\}$ and $D = \{1, 2, 3, 4, \ldots\}$. Then $C \subset D$ and $D \subset C$, and $C = D$.

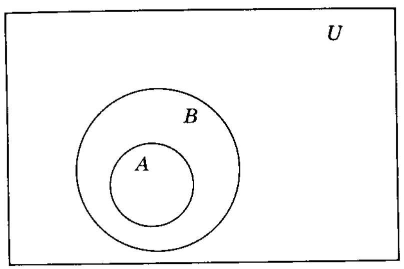


Fig. 1.2. $A \subset B$.

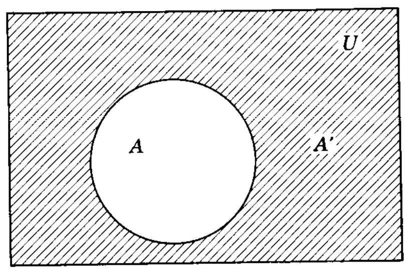


Fig. 1.3.

When A is a subset of a universal set U, it is natural to think of the set composed of all elements of U that are *not in A*. This set is called the *complement of A* and is denoted by A'. Thus, if U represents the set of integers and A the set of negative integers, then A' is the set of nonnegative integers, i.e., $A' = \{0, 1, 2, 3, \ldots\}$. The shaded area of Fig. 1.3 illustrates A'.

1.5. Operations on sets. We shall next discuss two methods for generating new sets from given sets.

Definition: The *intersection* of two sets P and Q is the set of all elements that belong to both P and Q.

The intersection of sets P and Q is symbolized by $P \cap Q$ and is read "P intersection Q" or "P cap Q."

Illustrations:

(*a*) If $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 4, 6, 8, 10\}$, then $A \cap B = \{2, 4\}$.
(*b*) If $D = \{1, 3, 5, \ldots\}$ and $E = \{2, 4, 6, \ldots\}$, then $D \cap E = \emptyset$.

When two sets have no elements in common they are said to be *disjoint sets* or *mutually exclusive sets*.

(*c*) If $F = \{0, 1, 2, 3, \ldots\}$ and $G = \{0, -2, -4, -6, \ldots\}$, then $F \cap G = \{0\}$.
(*d*) Given A is the set of all bachelors and B is the set of all males. Then $A \cap B = A$. Here A is a subset of B.

Care should be taken to distinguish between the set whose sole member is the number zero and the null set (see *b* and *c* above). They have quite distinct and different meanings. Thus $\{0\} \neq \emptyset$. The null set is empty of any elements. Zero is a number and can be a member of a set. The null set is a subset of all sets.

The intersection of two sets can be illustrated by a Venn diagram. The shaded area of Fig. 1.4 represents $A \cap B$.

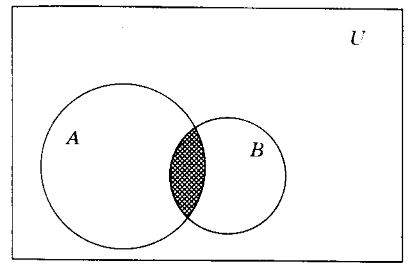


Fig. 1.4. $A \cap B$.

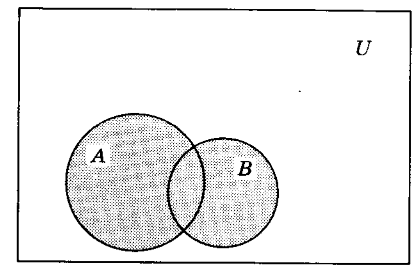


Fig. 1.5. $A \cup B$.

Definition: The *union* of two sets P and Q is the set of all elements that belong to either P or Q or that belong to both P and Q.

The union of sets P and Q is symbolized by $P \cup Q$ and is read "P union Q" or "P cup Q." The shaded area of Fig. 1.5 represents the Venn diagram of $A \cup B$.

Illustrations:

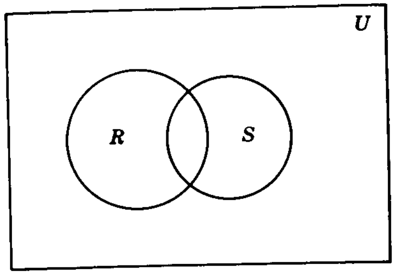
(*a*) If $A = \{1, 2, 3\}$ and $B = \{1, 3, 5, 7\}$, then $A \cup B = \{1, 2, 3, 5, 7\}$.
Note. Individual elements of the union are listed only once.
(*b*) If $A = \{\text{whole even numbers between } 2\frac{1}{2} \text{ and } 5\}$ and $B = \{\text{whole numbers between } 3\frac{1}{4} \text{ and } 6\frac{1}{2}\}$, then $A \cup B = \{4, 5, 6\}$ and $A \cap B = \{4\}$.
(*c*) If $P = \{\text{all bachelors}\}$ and $Q = \{\text{all men}\}$, then $P \cup Q = Q$.

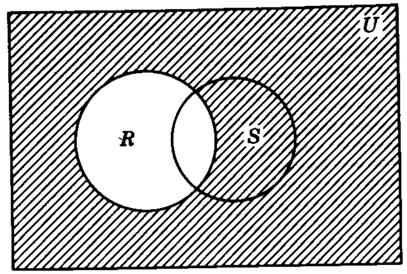
Example. Draw a Venn diagram to illustrate $(R' \cap S')'$ in the figure.

Solution

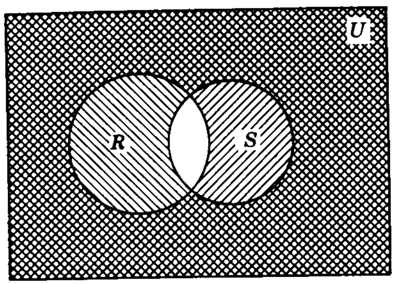
(*a*) Shade R'.
(*b*) Add a shade for S'.

$R' \cap S'$ is represented by the region common to the area slashed up to the

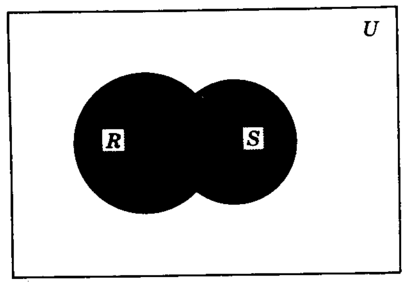




(a) R′



(b) R′ ∩ S′



(c) (R′ ∩ S′)′

right and the area slashed down to the right. $(R' \cap S')'$ is all the area in U that is not in $R' \cap S'$.

(*c*) The solution is shaded in the last figure.
We note that $(R' \cap S')' = R \cup S$.

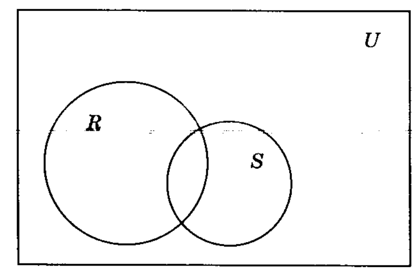
Exercises

1. Let $A = \{2, 3, 5, 6, 7, 9\}$ and $B = \{3, 4, 6, 8, 9, 10\}$.
   (*a*) What is $A \cap B$? (*b*) What is $A \cup B$?
2. Let $R = \{1, 3, 5, 7, \ldots\}$ and $S = \{0, 2, 4, 6, \ldots\}$.
   (*a*) What is $R \cap S$? (*b*) What is $R \cup S$?
3. Let $P = \{1, 2, 3, 4, \ldots\}$ and $Q = \{3, 6, 9, 12, \ldots\}$.
   (*a*) What is $P \cap Q$? (*b*) What is $P \cup Q$?
4. $(\{1, 3, 5, 7, 9\} \cap \{2, 3, 4, 5\}) \cup \{2, 4, 6, 8\} = ?$
5. Simplify: $\{4, 7, 8, 9\} \cup (\{1, 2, 3, \ldots\} \cap \{2, 4, 6, \ldots\})$.
6. Consider the following sets.
   A = {students in your geometry class}.
   B = {male students in your geometry class}.
   C = {female students in your geometry class}.
   D = {members of student body of your school}.
   What are (*a*) $A \cap B$; (*b*) $A \cup B$; (*c*) $B \cap C$; (*d*) $B \cup C$; (*e*) $A \cap D$; (*f*) $A \cup D$?

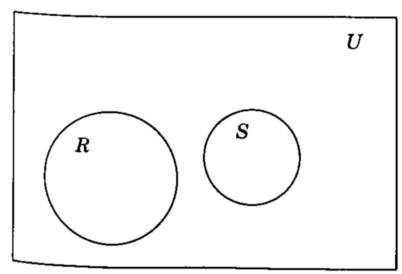
7. In the following statements P and Q represent sets. Indicate which of the following statements are true and which ones are false.
   (*a*) $P \cap Q$ is always contained in P.
   (*b*) $P \cup Q$ is always contained in Q.
   (*c*) P is always contained in $P \cup Q$.
   (*d*) Q is always contained in $P \cup Q$.
   (*e*) $P \cup Q$ is always contained in P.
   (*f*) $P \cap Q$ is always contained in Q.
   (*g*) P is always contained in $P \cap Q$.
   (*h*) Q is always contained in $P \cap Q$.
   (*i*) If $P \supset Q$, then $P \cap Q = P$.
   (*j*) If $P \supset Q$, then $P \cap Q = Q$.
   (*k*) If $P \subset Q$, then $P \cup Q = P$.
   (*l*) If $P \subset Q$, then $P \cup Q = Q$.
8. What is the solution set for the statement $a + 2 = 2$, i.e., the set of all solutions, of statement $a + 2 = 2$?
9. What is the solution set for the statement $a + 2 = a + 4$?
10. Let D be the set of ordered pairs (x, y) for which $x + y = 5$, and let E be the set of ordered pairs (x, y) for which $x - y = 1$. What is $D \cap E$?

11–30. Copy figures and use shading to illustrate the following sets.

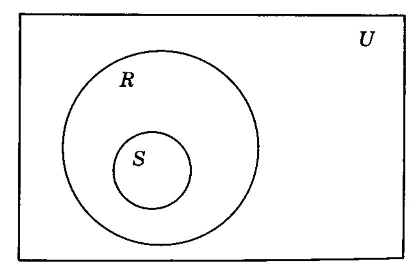
11. $R \cup S$.	12. $R \cap S$.
13. $(R \cap S)'$.	14. $(R \cup S)'$.
15. R'.	16. S'.
17. $(R')'$.	18. $R' \cup S'$.
19. $R' \cap S'$.	20. $(R' \cap S')'$.
21. $R \cup S$.	22. $R \cap S$.
23. $R' \cap S'$.	24. $R' \cup S'$.
25. $R \cup S$.	26. $R \cap S$.
27. $R' \cap S'$.	28. $R' \cup S'$.
29. $R' \cup S$.	30. $R \cup S'$.



Exs. 11–20.



Exs. 21–24.



Exs. 25–30.

1.6. Need for definitions. In studying geometry we learn to prove statements by a process of deductive reasoning. We learn to analyze a problem in terms of what data are given, what laws and principles may be accepted as true and, by careful, logical, and accurate thinking, we learn to select a solution to the problem. But before a statement in geometry can be proved, we must agree on certain definitions and properties of geometric figures. It is necessary that the terms we use in geometric proofs have exactly the same meaning to each of us.

Most of us do not reflect on the meanings of words we hear or read during the course of a day. Yet, often, a more critical reflection might cause us to wonder what really we have heard or read.

A common cause for misunderstanding and argument, not only in geometry but in all walks of life, is the fact that the same word may have different meanings to different people.

What characteristics does a good definition have? When can we be certain the definition is a good one? No one person can establish that his definition for a given word is a correct one. What is important is that the people participating in a given discussion agree on the meanings of the word in question and, once they have reached an understanding, no one of the group may change the definition of the word without notifying the others.

This will especially be true in this course. Once we agree on a definition stated in this text, we cannot change it to suit ourselves. On the other hand, there is nothing sacred about the definitions that will follow. They might well be improved on, as long as everyone who uses them in this text agrees to it.

A good definition in geometry has two important properties:

1. The words in the definition must be simpler than the word being defined and must be clearly understood.

2. The definition must be a reversible statement.

Thus, for example, if "right angle" is defined as "an angle whose measure is 90," it is assumed that the meaning of each term in the definition is clear and that:

1. If we have a right angle, we have an angle whose measure is 90.

2. Conversely, if we have an angle whose measure is 90, then we have a right angle.

Thus, the converse of a good definition is always true, although the converse of other statements are *not necessarily* true. The above statement and its converse can be written, "An angle is a right angle if, and only if, its measure is 90. The expression "if and only if" will be used so frequently in this text that we will use the abbreviation "iff" to stand for the entire phrase.

1.7. Need for undefined terms. There are many words in use today that are difficult to define. They can only be defined in terms of other equally undefinable concepts. For example, a "straight line" is often defined as a line "no part of which is curved." This definition will become clear if we can define the word curved. However, if the word curved is then defined as a line "no part of which is straight," we have no true understanding of the definition of the word "straight." Such definitions are called "*circular definitions*." If we define a straight line as one extending without change in direction, the word "direction" must be understood. In defining mathematical terms, we start with undefined terms and employ as few as possible of those terms that are in daily use and have a common meaning to the reader.

In using an undefined term, it is assumed that the word is so elementary that its meaning is known to all. Since there are no easier words to define the term, no effort is made to define it. The dictionary must often resort to "defining" a word by either listing other words, called *synonyms*, which have the same (or almost the same) meaning as the word being defined or by describing the word.

We will use **three undefined geometric terms** in this book. They are: **point, straight line, and plane.** We will resort to synonyms and descriptions of these words in helping the student to understand them.

1.8. Points and lines. Before we can discuss the various geometric figures as sets of points, we will need to consider the nature of a point. What is a point? Everyone has some understanding of the term. Although we can represent a point by marking a small dot on a sheet of paper or on a blackboard, it certainly is not a point. If it were possible to subdivide the marker, then subdivide again the smaller dots, and so on indefinitely, we still would not have a point. We would, however, approach a condition which most of us assign to that of a point. Euclid attempted to do this by defining a point as that which has position but no dimension. However, the words "position" and "dimension" are also basic concepts and can only be described by using circular definitions.

We name a point by a capital letter printed beside it, as point "A" in Fig. 1.6. Other geometric figures can be defined in terms of sets of points which satisfy certain restricting conditions.

We are all familiar with lines, but no one has seen one. Just as we can represent a point by a marker or dot, we can represent a line by moving the tip of a sharpened pencil across a piece of paper. This will produce an approximation for the meaning given to the word "line." Euclid attempted to define a line as that which has only one dimension. Here, again, he used

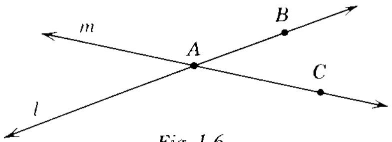


Fig. 1.6.

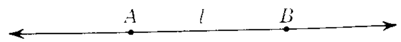


Fig. 1.7.

the undefined word "dimension" in his definition. Although we cannot define the word "line," we recognize it as a set of points.

On page 11, we discussed a "straight line" as one no part of which is "curved," or as one which extends without change in directions. The failures of these attempts should be evident. However, the word "straight" is an abstraction that is generally used and commonly understood as a result of many observations of physical objects. The line is named by labeling two points on it with capital letters or by one lower case letter near it. The straight line in Fig. 1.7 is read "line AB" or "line l." Line AB is often written "$\overleftrightarrow{AB}$." In this book, unless otherwise stated, when we use the term "line," we will have in mind the concept of a *straight* line.

If $B \in l$, $A \in l$, and $A \neq B$, we say that l is the line which *contains* A and B. Two points determine a line (see Fig. 1.7). Thus $\overleftrightarrow{AB} = \overleftrightarrow{BA}$.

Two straight lines can intersect in only one point. In Fig. 1.6, $\overleftrightarrow{AB} \cap \overleftrightarrow{AC} = \{A\}$. What is $\overleftrightarrow{AB} \cap \overleftrightarrow{BC}$?

If we mark three points R, S, and T (Fig. 1.8) all on the same line, we see that $\overleftrightarrow{RS} = \overleftrightarrow{ST}$. Three or more points are ***collinear*** iff they belong to the same line.

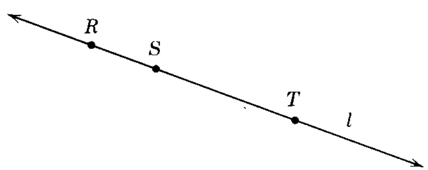


Fig. 1.8.

1.9. Solids and planes. Common examples of solids are shown in Fig. 1.9.

The geometric solid shown in Fig. 1.10 has six faces which are smooth and flat. These faces are subsets of *plane surfaces* or simply *planes*. The surface of a blackboard or of a table top is an example of a plane surface. A plane can be thought of as a set of points.

Definition. A set of points, all of which lie in the same plane, are said to be *coplanar*. Points D, C, and E of Fig. 1.10 are coplanar. A plane can be named by using two points or a single point in the plane. Thus, Fig. 1.11

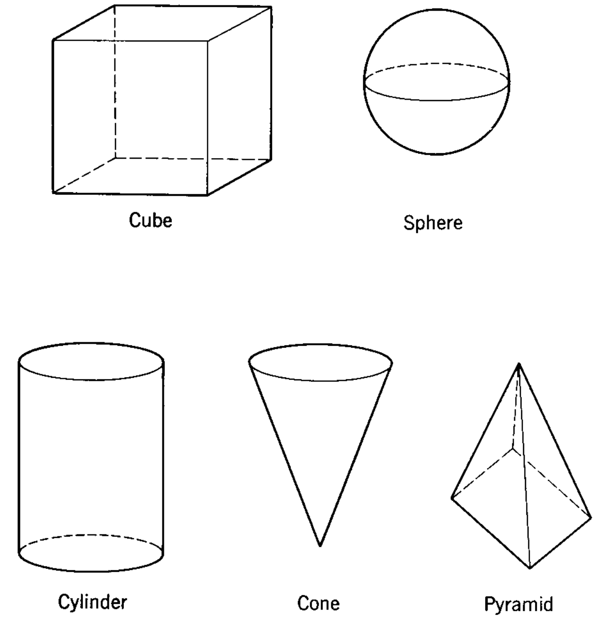


Fig. 1.9.

represents plane MN or plane M. We can think of the plane as being made up of an infinite number of points to form a surface possessing no thickness but having infinite length and width.

Two lines lying in the same plane whose intersection is the null set are said to be ***parallel lines***. If line l is parallel to line m, then $l \cap m = \emptyset$. In Fig. 1.10, $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{DC}$ and $\overleftrightarrow{AD}$ is parallel to $\overleftrightarrow{BC}$.

The drawings of Fig. 1.12 and Fig. 1.13 illustrate various combinations of points, lines, and planes.

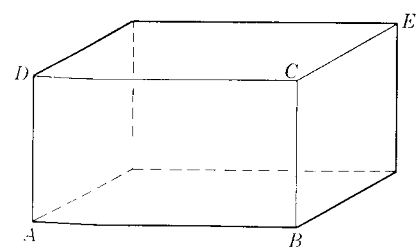


Fig. 1.10.

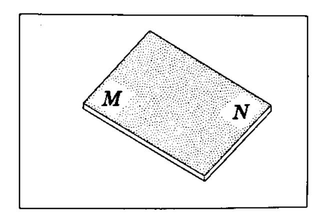


Fig. 1.11.

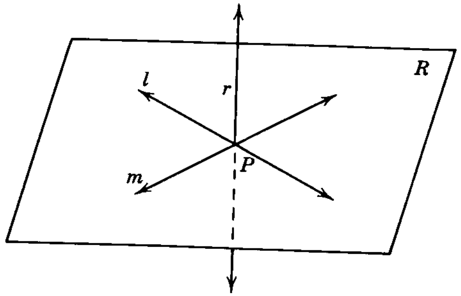


Fig. 1.12.

Line r intersects plane R.
Plane R contains line l and m.
Plane R passes through lines l and m.
Plane R does not pass through line r.

Plane MN and plane RS intersect in $\overleftrightarrow{AB}$.
Plane MN and plane RS both pass through $\overleftrightarrow{AB}$.
$\overleftrightarrow{AB}$ *lies in both planes.*
$\overleftrightarrow{AB}$ *is contained in planes MN and RS.*

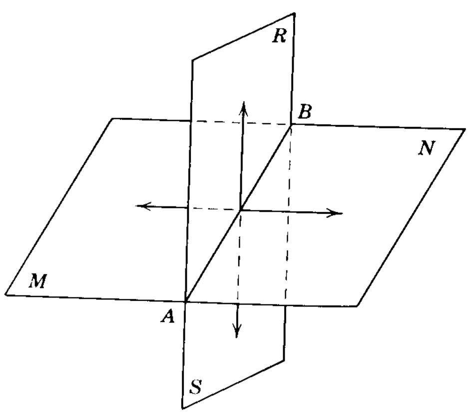


Fig. 1.13.

Exercises

1. How many points does a line contain?
2. How many lines can pass through a given point?
3. How many lines can be passed through two distinct points?
4. How many planes can be passed through two distinct points?

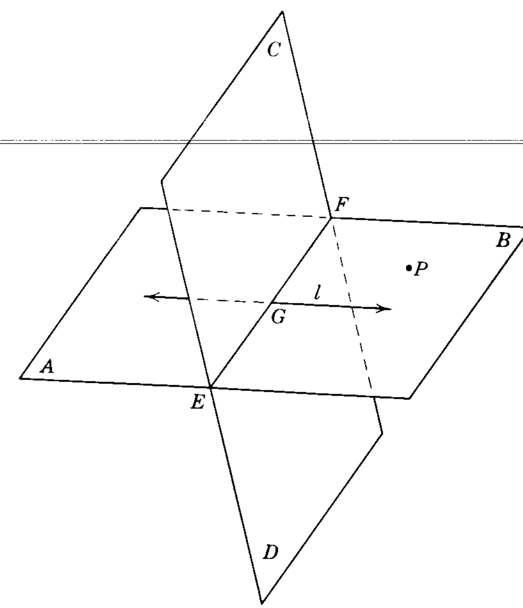
5. Can a line always be passed through any three distinct points?
6. Can a plane always be passed through any three distinct points?
7. Can two planes ever intersect in a single point?
8. Can three planes intersect in the same straight line?

9–17. Refer to the figure and indicate which of the following statements are true and which are false.

9. Plane AB intersects plane CD in line l.
10. Plane AB passes through line l.
11. Plane AB passes through $\overleftrightarrow{EF}$.
12. Plane CD passes through $\overleftrightarrow{EF}$.
13. $P \in$ plane CD.
14. (plane AB) $\cap$ (plane CD) $= \overleftrightarrow{EF}$.
15. $l \cap \overleftrightarrow{EF} = G$.
16. (plane CD) $\cap$ $l = G$.
17. (plane AB) $\cap$ $\overleftrightarrow{EF} = \overleftrightarrow{EF}$.

18–38. Draw pictures (if possible) that illustrate the situations described.

18. l and m are two lines and $l \cap m = \{P\}$.
19. l and m are two lines, $P \in l, R \in l, S \in m$ and $\overleftrightarrow{RS} \neq \overleftrightarrow{PR}$.
20. $C \notin \overleftrightarrow{AB}$, and $A \neq B$.
21. $R \in \overleftrightarrow{ST}$.



Exs. 9–17.

22. r and s are two lines, and $r \cap s = \emptyset$.
23. r and s are two lines, and $r \cap s \neq \emptyset$.
24. $P \notin \overleftrightarrow{KL}$, $P \in l$, and $l \cap \overleftrightarrow{KL} = \emptyset$.
25. R, S, and T are three points and $T \in (\overleftrightarrow{RT} \cap \overleftrightarrow{ST})$.
26. r and s are two lines, $A \neq B$, and $\{A, B\} \subset (r \cap s)$.
27. P, Q, R, and S are four points, $Q \in \overleftrightarrow{PR}$, and $R \in \overleftrightarrow{QS}$.
28. P, Q, R, and S are four noncollinear points, $Q \in \overleftrightarrow{PR}$, and $Q \in \overleftrightarrow{PS}$.
29. A, B, and C are three noncollinear points, A, B, and D are three collinear points, and A, C, and D are three collinear points.
30. l, m, and n are three lines, and $P \in (m \cap n) \cap l$.
31. l, m, and n are three lines, $A \neq B$, and $\{A, B\} \subset (l \cap m) \cap n$.
32. l, m, and n are three lines, $A \neq B$, and $\{A, B\} = (l \cap m) \cup (n \cap m)$.
33. A, B, and C are three collinear points, C, D, and E are three noncollinear points, and $E \in \overleftrightarrow{AB}$.
34. (plane RS) $\cap$ (plane MN) $= \overleftrightarrow{AB}$.
35. (plane AB) $\cap$ (plane CD) $= \emptyset$.
36. line $l \subset$ plane AB. line $m \subset$ plane CD. $l \cap m = \{P\}$.
37. (plane AB) $\cap$ (plane CD) $= l$. line $m \in$ plane CD. $l \cap m = \emptyset$.
38. (plane AB) $\cap$ (plane CD) $= l$. line $m \in$ plane CD. $l \cap m \neq \emptyset$.

1.10. Real numbers and the number line. The first numbers a child learns are the counting or natural numbers, e.g., $\{1, 2, 3, \ldots\}$. The natural numbers are infinite; that is, given any number, however large, there is always another number larger (add 1 to the given number). These numbers can be represented by points on a line. Place a point O on the line $X'X$ (Fig. 1.14). The point O will divide the line into two parts. Next, let A be a point on $X'X$ to the right of O. Then, to the right of A, mark off equally spaced points B, C, D, ... For every positive whole number there will be exactly one point to the right of point O. Conversely, each of these points will represent only one positive whole number.

In like manner, points $R, S, T, \ldots$ can be marked off to the left of point O to represent negative whole numbers.

The distance between points representing consecutive integers can be divided into halves, thirds, fourths, and so on, indefinitely. Repeated division would make it possible to represent all positive and negative fractions with points on the line. Note Fig. 1.15 for a few of the numbers that might be assigned to points on the line.

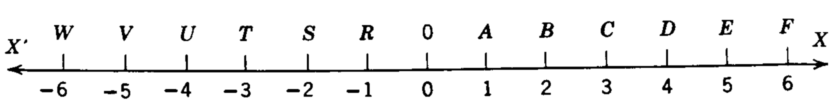


Fig. 1.14.

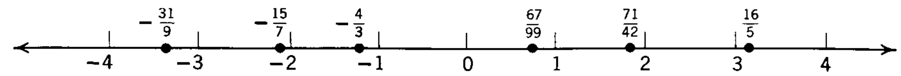


Fig. 1.15.

We have now expanded the points on the line to represent all real rational numbers.

Definition: A *rational number* is one that can be expressed as a quotient of integers.

It can be shown that every quotient of two integers can be expressed as a repeating decimal or decimal that terminates, and every such decimal can be written as an equivalent indicated quotient of two integers. For example, $13/27 = 0.481481\ldots$ and $1.571428571428\ldots = 11/7$ are rational numbers.

The rational numbers form a very large set, for between any two rational numbers there is a third one. Therefore, there are an infinite number of points representing rational numbers on any given scaled line. However, the rational numbers still do not completely fill the scaled line.

Definition: An *irrational number* is one that cannot be expressed as the quotient of two integers (or as a repeating or terminating decimal).

Examples of irrational numbers are $\sqrt{2}, -\sqrt{3}, \sqrt[3]{5}$, and π. Approximate locations of some rational and irrational numbers on a scaled line are shown in Fig. 1.16.

The union of the sets of rational and irrational numbers form the set of *real numbers*. The line that represents all the real numbers is called the *real number line*. The number that is paired with a point on the number line is called the *coordinate* of that point.

We summarize by stating that the real number line is made up of an infinite set of points that have the following characteristics.

1. *Every point on the line is paired with exactly one real number.*
2. *Every real number can be paired with exactly one point on the line.*

When, given two sets, it is possible to pair each element of each set with exactly one element of the other, the two sets are said to have a *one-to-one correspondence*. We have just shown that there is a one-to-one correspondence between the set of real numbers and the set of points on a line.

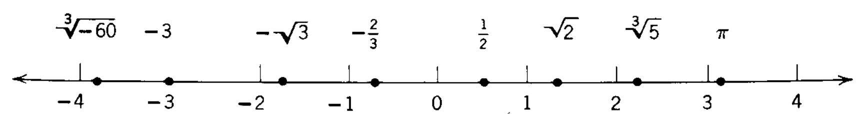


Fig. 1.16.

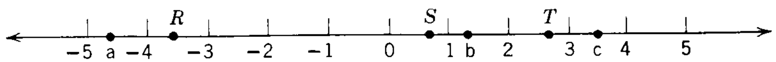


Fig. 1.17.

1.11. Order and the number line. All of us at one time or another engage in comparing sizes of real numbers. Symbols are often used to indicate the relative sizes of real numbers. Consider the following.

Symbol	*Meaning*
$a = b$	a equals b
$a \neq b$	a is not equal to b
$a > b$	a is greater than b
$a < b$	a is less than b
$a \geqslant b$	a is either greater than b or a is equal to b
$a \leqslant b$	a is either less than b or a is equal to b

It should be noted that $a > b$ and $b < a$ have exactly the same meaning; that is, if a is more than b, then b is less than a.

The number line is a convenient device for visualizing the *ordering* of real numbers. If $b > a$, the point representing the number b will be located to the right of the point on the number line representing the number a (see Fig. 1.17). Conversely, if point S is to the right of point R, then the number which is assigned to S must be larger than that assigned to R. In the figure, $b < c$ and $c > a$.

When we write or state $a = b$ we mean simply that a and b are different names for the same number. Thus, points which represent the same number on a number line must be identical.

1.12. Distance between points. Often in the study of geometry, we will be concerned with the "distance between two points." Consider the number line of Fig. 1.18 where points A, P, B, C, respectively represent the integers $-3, 0, 3, 6$. We note that A and B are the same distance from P, namely 3.

Next consider the distance between B and C. While the coordinates differ in these and the previous two cases, it is evident that the distance between the points is represented by the number 3.

How can we arrive at a rule for determining distance between two points? We could find the distance between two points on a scaled line by subtracting

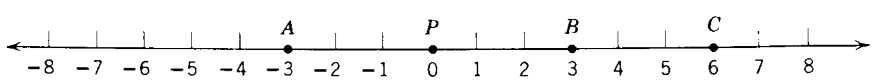


Fig. 1.18.

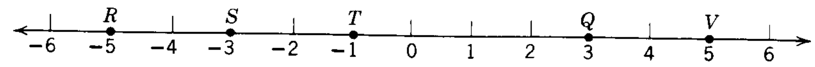


Fig. 1.19.

the smaller number represented by these two points from the larger. Thus, in Fig. 1.19:

The distance from T to V $= 5 - (-1) = 6$.
The distance from S to T $= (-1) - (-3) = 2$.
The distance from Q to R $= 3 - (-5) = 8$.

Another way we could state the above rule could be: "Subtract the coordinate of the left point from that of the point to the right." However, this rule would be difficult to apply if the coordinates were expressed by place holders a and b. We will need to find some way of always arriving at a number that is positive and is associated with the difference of the coordinates of the point. To do this we use the symbol $|\ |$. The symbol $|x|$ is called the *absolute value of x*. In the study of algebra the absolute value of any number x is defined as follows.

$$|x| = x \text{ if } x \geqslant 0$$
$$|x| = -x \text{ if } x < 0$$

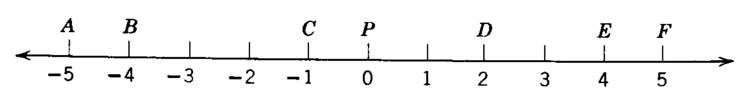
Consider the following illustrations of the previous examples.

Column 1	*Column 2*
$\lvert 3\rvert = 3$	$\lvert -3\rvert = 3$
$\lvert 5-(-1)\rvert = \lvert 6\rvert = 6$	$\lvert (-1)-(+5)\rvert = \lvert -6\rvert = 6$
$\lvert (-1)-(-3)\rvert = \lvert 2\rvert = 2$	$\lvert (-3)-(-1)\rvert = \lvert -2\rvert = 2$
$\lvert 3-(-5)\rvert = \lvert 8\rvert = 8$	$\lvert (-5)-(+3)\rvert = \lvert -8\rvert = 8$

Thus, we note that to find the distance between two points we need only to subtract the coordinates in either order and then take the absolute value of the difference. *If a and b are the coordinates of two points, the distance between the points can be* expressed either by $|a-b|$ or $|b-a|$.

Exercises

1. What is the coordinate of B? of D?
2. What point lies halfway between B and D?
3. What is the coordinate of the point 7 units to the left of D?



Exs. 1–17.

4. What is the coordinate of the point 3 units to the right of C?
5. What is the coordinate of the point midway between C and F?
6. What is the coordinate of the point midway between D and F?
7. What is the coordinate of the point midway between C and E?
8. What is the coordinate of the point midway between A and C?

9–16. Let a, b, c, d, e, f, p represent the coordinates of points A, B, C, D, E, F, P, respectively. Determine the values of the following.

9. $e-p$	10. $b-p$	11. $b-c$
12. $\|d-b\|$	13. $\|e-d\|$	14. $\|d-f\|$
15. $\|c-d\|$	16. $\|a-c\|$	17. $\|a-e\|$

18–26. Evaluate the following.

18. $\|-1\|+\|2\|$	19. $\|-3\|+\|-4\|$	20. $\|-8\|-\|-3\|$
21. $\|-4\|-\|-6\|$	22. $\|-3\|\times\|3\|$	23. $2\|-4\|$
24. $\|-4\|^2$	25. $\|2\|^2+\|-2\|^2$	26. $\|2\|^2-\|-2\|^2$

1.13. Segments. Half-lines. Rays. Let us next consider that part of the line between two points on a line.

Definitions: The part of line AB between A and B, together with points A and B, is called *segment AB* (Fig. 1.19*a*). Symbolically it is written $\overline{AB}$. The points A and B are called the *endpoints* of $\overline{AB}$. The number that tells how far it is from A to B is called the *measure* (or *length*) of $\overline{AB}$. In this text we will use the symbol $m\overline{AB}$ to mean the length of $\overline{AB}$.

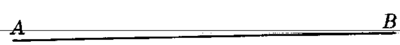


Fig. 1.19a. Segment AB.

The student should be careful to recognize the differences between the meanings of the symbols $\overline{AB}$ and $m\overline{AB}$. The first refers to a geometric figure; the second to a number.

Definition: B is *between* A and C (see Fig. 1.20) if, and only if, A, B, and C are distinct points on the same line and $m\overline{AB}+m\overline{BC}=m\overline{AC}$. Using the equal sign implies simply that the name used on the left ($m\overline{AB}+m\overline{BC}$) and the name used on the right of the equality sign ($m\overline{AC}$) are but two different names for the same number.

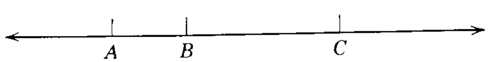


Fig. 1.20. $m\overline{AB}+m\overline{BC}=m\overline{AC}$.

Definition: A point B is the *midpoint* of $\overline{AC}$ iff B is between A and C and $m\overline{AB}=m\overline{BC}$. The midpoint is said to *bisect* the segment (see Fig. 1.21).

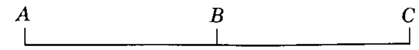


Fig. 1.21. $m\overline{AB}=m\overline{BC}$.

A line or a segment which passes through the midpoint of a second segment bisects the segment. If, in Fig. 1.22, M is the midpoint of $\overline{AB}$, then $\overleftrightarrow{CD}$ bisects $\overline{AB}$.

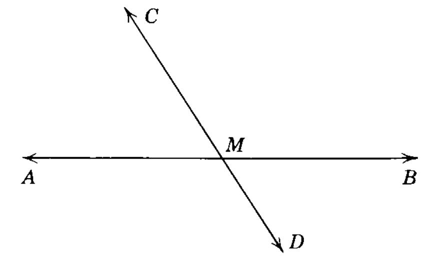


Fig. 1.22.

Definition: The set consisting of the points between A and B is called an *open segment* or the *interval* joining A and B. It is designated by the symbol $\overset{\circ\!-\!\circ}{AB}$.

Definition: For any two distinct points A and B, the figure $\{A\}\cup\{\overset{\circ\!-\!\circ}{AB}\}$ is called a *half-open segment*. It is designated by the symbol $\overset{-\circ}{AB}$. Open segments and half-open segments are illustrated in Fig. 1.23.

Every point on a line divides that line into two parts. Consider the line l through points A and B (Fig. 1.24*a*).

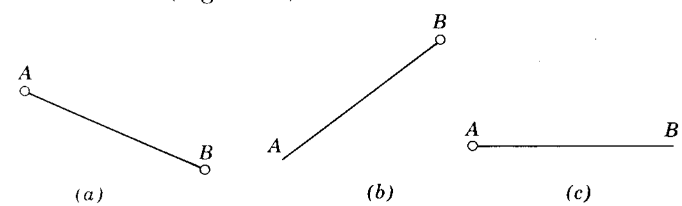


Fig. 1.23. (a) $\overset{\circ\!-\!\circ}{AB}$ (b) $\overset{-\circ}{AB}$ (c) $\overset{\circ-}{AB}$.

Definition: If A and B are points of line l, then the set of points of l which are on the same side of A as is B is the *half-line from A through B* (Fig. 1.24*b*).

The symbol for the half-line from A through B is $\overset{\circ\!\rightarrow}{AB}$ and is read "half-line AB." The arrowhead indicates that the half-line includes *all* points of the line on the same side of A as is B. The symbol for the half-line from B through A (Fig. 1.24*c*) is $\overset{\circ\!\rightarrow}{BA}$. Note that A is not an element of $\overset{\circ\!\rightarrow}{AB}$. Similarly, B does not belong to $\overset{\circ\!\rightarrow}{BA}$.