6

Time Value of Money

When a person loans money, a charge is made for the use of these borrowed funds. The lender perhaps could have invested the funds somewhere else and made a profit; therefore, the interest is the compensation for the foregone profit. The borrower may look upon this interest as the cost of renting money. The amount charged depends on the scarcity of money, the size of the loan, the length of the loan, the risk the lender feels that the loan may not be paid back, and the prevailing economic conditions. Because engineers may be involved in the presentation and/or the evaluation of an investment of money in a venture, it is important that they understand the time value of money and how it is applied in the evaluation of projects. Later we shall see that in modern times, the term "return on investment" is used for the classical term "interest." It is the charge paid for borrowed money.

6.1 INTEREST RATE

The interest rate is the ratio of the interest charged at the end of a period (usually 1 year) to the amount of money owed at the beginning of the period expressed as a percentage. For example, if $10 of interest is payable at the end of a year on a loan of $100, the interest rate is $10/$100 or 0.10 or 10% interest. When an interest rate is quoted, it is usually expressed on an annual basis unless otherwise quoted. The nominal interest is 10% without any consideration of the effect of compounding during the year. Interest may be compounded on bases other than annual and this topic will be considered later in this chapter [1].

6.2 INTEREST NOMENCLATURE

The nomenclature used in interest calculations may be found in Table 6.1. The terminology presented in the table is to acquaint the reader with the numerous meanings for the same symbols and will be used throughout the text.

6.3 SIMPLE INTEREST

If P is the principal, the loan amount or the original capital, n is the number of interest periods, and i is then interest rate for the period, the amount of simple interest I earned for n periods is

$$I = Pin \tag{6.1}$$

Ultimately, the principal must be paid plus the simple interest for n periods at a future time, F; therefore,

$$F = P + I = P + Pin = P(1 + in) \tag{6.2}$$

The interest is charged on the original loan and not on the unpaid balance. Simple interest is paid at the end of each time interval. Although the simple interest concept still exists, it is seldom used [1].

Example 6.1

If $1000 has been borrowed at 10% simple interest for 4 years, develop a table of values for the interest owed each year and the total amount owed.

Table 6.2 is the solution of the simple interest calculation in Example 6.1.

TABLE 6.1 Interest Nomenclature

Symbol	Definition
F	Future sum
	Future value
	Future worth
	Future amount
P	Principal
	Present worth
	Present value
	Present amount
A	End of period payment in a uniform series

TABLE 6.2 10% Simple Interest on $1000

End of year	Interest owed at end of each year	Total amount owed
0	—	$1000
1	$100	1100
2	100	1200
3	100	1300
4	100	1400

6.4 COMPOUND INTEREST

Since interest has a time value, often the lender will invest this interest and earn more additional interest [1]. It is assumed that the interest is not withdrawn but is added to the principal and then in the next period interest is calculated based upon the principal plus the interest in the preceding period. This is called compound interest. In equation format, the future amount for each year is

$$\text{Year 1: } P + Pi = P(1+i) = F_1$$

$$\text{Year 2: } P + Pi(1+i) = P(1+i)^2 = F_2$$

$$\text{Year } n\text{: } P(1+i)^n = F_n$$

So, the principal plus interest after n years is

$$F = P(1+i)^n \tag{6.3}$$

An interest rate is quoted on an annual basis and is referred to as nominal interest. However, interest may be payable on a semiannual, quarterly, monthly, or daily basis. In order to determine the amount compounded, the following equation applies:

$$F = P\left[1 + \left(\frac{i}{m}\right)\right]^{(m)(n)} \tag{6.4}$$

where

m = the number of interest periods per year
n = the number of years
i = the nominal interest

Example 6.2

If \$2500 were invested for 5 years at 10% nominal interest compounded quarterly, what would be the future amount?

$$F = P\left[1 + \left(\frac{i}{m}\right)\right]^{(m)(n)} = (\$2500)\left[\left(1 + \frac{0.10}{4}\right)^{(5)(4)}\right]$$

$$F = (\$2500)(1.025)^{20} = (\$2500)(1.638616) = \$4096.54$$

In this example, if the interest were compounded monthly, the future amount would be

$$F = P\left[1 + \left(\frac{0.1}{12}\right)\right]^{(12)(5)} = (\$2500)(1.00833)^{(60)}$$

$$F = (\$2500)(1.645276) = \$4113.19$$

If the interest had been compounded on a daily basis, then

$$F = P\left[1 + \left(\frac{0.10}{365}\right)\right]^{(365)(5)} = (\$2500)(1.000274)^{(1825)}$$

$$F = (\$2500)(1.648690) = \$4121.73$$

(*Note*: It is necessary to carry the number of digits to six places to reflect the difference in the future amounts. See also Sec. 6.8).

If companies immediately invest receipts from sales and/or services, the interest on this cash is compounded and the process is called continuous compounding. This will be discussed later in this chapter.

6.5 COMPOUND INTEREST FACTORS

6.5.1 Compounding–Discounting

The process of moving money forward in time is referred to as *compounding*. Moving money backward in time is called *discounting*. These processes are shown in Figure 6.1.

The calculations associated with these processes are not difficult but as the number of compounding or discounting periods increases, the calculations may be time consuming. The periods are in years and the interest is normally on an annual basis; however, shorter compounding periods may be used as mentioned previously [1]. Only end-of-year cash flows will be considered, a widely used convention.

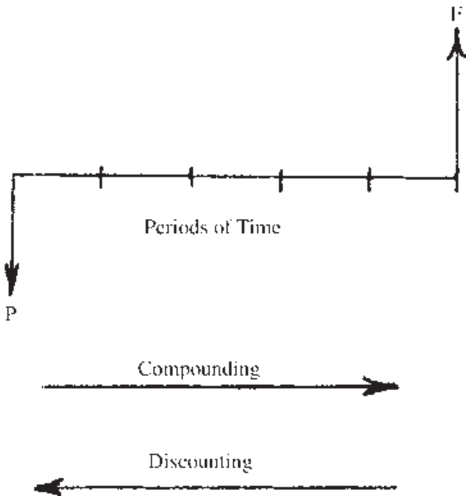

FIGURE 6.1 Compounding–discounting diagram.

6.5.2 Compound Interest Factors

6.5.2.1 Single-Payment Compound Amount Factor

Let's assume that $1000 is borrowed at 10% interest compounded annually for 5 years. Table 6.3 is a summary of the results.

An equation for this process may be written as follows:

$$F = P(1 + i)^n \tag{6.5}$$

where

F = a future sum of money
P = a present sum of money
i = interest rate per period expressed as a decimal
n = number of interest periods

At the end of 5 years the future amount is

$$F = (\$1000)(1 + 0.10)^5 = (\$1000)(1.61051) = \$1610.51$$

The term $(1 + i)^n$ is known as the single-payment compound amount factor [1]. A short-hand designation for the equation is $(F/P\ i,n) = (1 + i)^n$. Tables of these values may be found in economic texts; however, it is a simple matter to determine the factor without extensive tables by using hand-held calculators or by computers. Therefore, such tables will not be included in this text.

TABLE 6.3 10% Compound Interest on $1000

Year	Amount owed at beginning of year, $	Interest accrued during year, $	Amount owed at end of year, $
1	1000.00	100.00	$1000 \times 1.10 = 1100.00$
2	1100.00	110.00	$1000 \times (1.10)^2 = 1210.00$
3	1210.00	121.00	$1000 \times (1.10)^3 = 1331.00$
4	1331.00	133.10	$1000 \times (1.10)^4 = 1464.10$
5	1464.10	146.41	$1000 \times (1.10)^5 = 1610.51$

Example 6.3

If $2500 were invested at 5% interest compounded annually, what would be the balance in the account after 5 years?

This is a simple problem but as with all these problems, as they become more complex, it is advisable to analyze the information for use in Eq. (6.3). For example,

$$F = ?$$

$$P = \$2500$$

$$i = 0.05$$

$$n = 5$$

A device useful in understanding these problems is the "time" or "dollar–time" diagram. In Figure 6.2, an arrow in the downward direction represents money invested at the present time, and an arrow in the upward direction is the balance at the end of the period including principal plus interest.

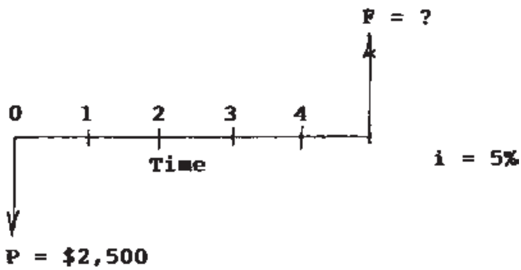

FIGURE 6.2 Dollar–time diagram for Example 6.3.

6.5.2.2 Single-Payment Present Worth Factor

This factor is the reciprocal of the single-payment compound amount factor [1]. Equation (6.3) is solved for P, the present amount instead of F.

$$P = \frac{F}{(1 + \mathrm{i})^n} \tag{6.6}$$

Example 6.4

IF \$5000 were needed 5 years from now to meet a certain obligation, how much would have to be deposited at 4% interest compounded annually to have \$5000 in 5 years?

Analyzing the problem,

$F = \$5000$

$P = ?$

$I = 0.04$

$n = 5$

$$P = \frac{\$5000}{(1.04)^5} = \frac{\$5000}{(1.2167)} = \$3946.33$$

The dollar–time diagram for Example 6.4 is found in Figure 6.3.

6.5.2.3 Uniform Series Compound Amount Factor

In economic analysis of alternative investments, a series of equal receipts or payments occurs at the end of a successive series of periods [1]. The equation is

$$F = A\left[\frac{(1 + i)^n - 1}{(i)}\right] \tag{6.7}$$

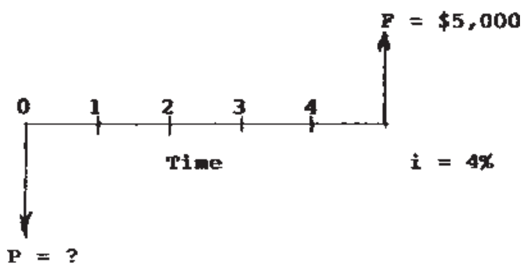

FIGURE 6.3 Dollar–time diagram for Example 6.4.

The bracketed term is the uniform series compound amount factor. Using the shorthand format,

$$F = A(F/A\ i,n)$$

Example 6.5 and the dollar–time diagram, Figure 6.4, may illustrate this process.

Example 6.5

If \$100 were deposited at the end of every year for 5 years in an account earning 6% interest compounded annually, how much will be in the account at the end of 5 years?

In the solution, F is the quantity required, $A = \$100$, $i = 0.06$, and $n = 5$.

$$F = (\$100)\left[\frac{1.06^5 - 1}{(0.06)}\right] = (\$100)\left[\frac{1.33822 - 1}{(0.06)}\right] = \left[\frac{(\$100)(0.33822)}{(0.06)}\right]$$

$$= \$563.70$$

The $(F/A\ i,n)$ factor is 5.6370.

The model assumes that we are waiting 1 year to start the annual deposits, it may not seem logical but you may recall that this is the assumption behind all these equations, namely the end-of-year convention. The assumption is that the money is accumulated during the year and moved to the end of the period. However, it may be desirable to start the payments at time 0, and this can be done by applying the single-payment compound amount factor to the calculation, moving all payments up 1 year. The result is

$$F = (\$100)(1.06)^1(5.6370) = \$597.52$$

This is an example of using the product of two factors to change the time base. Note that more money accumulates when the money is put to work earlier.

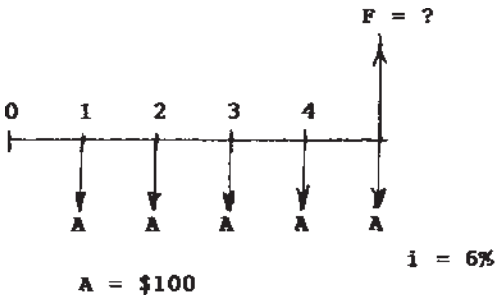

FIGURE 6.4 Dollar–time diagram for Example 6.5.

6.5.2.4 Uniform Series Sinking Fund Factor

If a uniform amount of money is placed in a fund, a sinking fund, at a certain interest rate for a number of years, a certain amount will be achieved in the future [1,2]. Equation (6.7) is solved for A instead of F as follows:

$$A = F\left[\frac{(i)}{(1+i)^n - 1}\right] \tag{6.8}$$

Example 6.6

Assuming there is a need for \$5000 in 5 years as in Example 6.4, it was decided to deposit a certain amount of money at the end of every year for 5 years at 4% interest instead of a single sum at time 0. What would the annual amount to be deposited be?

An analysis of the problem is that $F = \$5000$, $n = 5$ years, $i = 0.04$, and $A = ?$

$$A = F\{(i)[(1+i)^n - 1]\} = (\$5000)\left[\frac{(0.04)}{(1.04)^5 - 1}\right] = \frac{(\$5000)(0.04)}{(1.21665) - 1}$$

$$= \$923.15$$

Therefore, we have found that a present sum of \$3946.33 (Example 6.4), a uniform series of \$923.15, and a future sum of \$5000 are equivalent as long as $n = 5$ periods and the interest is 4% per period.

The dollar–time diagram is shown in Figure 6.5 for this model.

6.5.2.5 Uniform Series Present Worth Factor

This factor is the product of two factors that were presented previously, the uniform series compound amount factor and the single-payment present worth

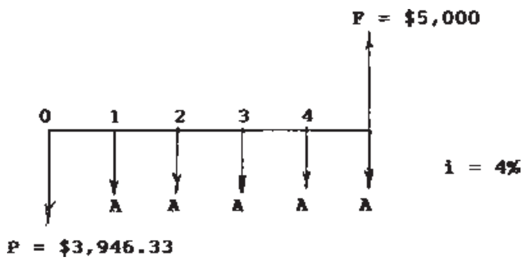

FIGURE 6.5 Dollar–time diagram for Example 6.6.

factor, $(F/A\ i,n)(P/F\ i,n)$ respectively [1].

$$P = A(P/A\, i,n) = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] \tag{6.9}$$

Example 6.7 is an illustration of how this factor is used and Figure 6.6 is the dollar–time diagram for this model.

Example 6.7

A person wants to borrow as much money as possible today with an annual payment of $1000 at the end of each year for 5 years. If he is charged 7% interest compounded annually, how much could he borrow?

In this problem $A = \$1000$, $i = 0.07$, $n = 5$, and $P = ?$

$$P = A\left[\frac{(1+i)^n - 1}{(i)(1+i)^n}\right] = (\$1000)\left[\frac{(1.07)^5 - 1}{(0.07)(1.07)^5}\right] = (\$1000)\left[\frac{(0.40255)}{(0.09818)}\right]$$

$$= \$4130.25$$

6.5.2.6 Uniform Series Capital Recovery Factor

This factor is used by a lender to determine the annual payments to be made by a borrower in order to recover the capital plus interest [1].

$$A = P\left[\frac{(i)(1+i)^n}{(1+i)^n - 1}\right] \tag{6.10}$$

The shorthand expression is

$$A = P(A/P\, i,n)$$

The following example illustrates how Eq. (6.8) is used.

Example 6.8

A person desires to borrow $18,500 now to be paid back in 10 years at 8.5% compounded annually. How much is this person required to pay annually?

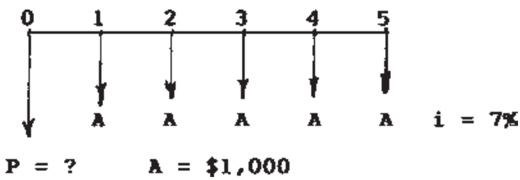

FIGURE 6.6 Dollar–time diagram for Example 6.7.

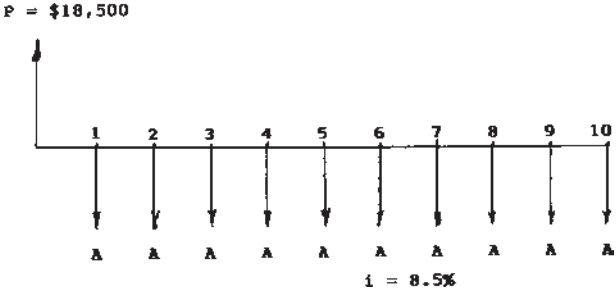

FIGURE 6.7 Dollar–time diagram for Example 6.8.

In this problem, $P = \$18{,}500$, $i = 0.085$, $n = 10$ years, and $A = ?$

$$A = P\left[\frac{(i)(1+i)^n}{(1+i)^n - 1}\right] = (\$18{,}500)\left[\frac{(0.085)(1.085)^{10}}{(1.085)^{10} - 1}\right]$$

$$= (\$18{,}500)\left[\frac{(0.085)(2.26098)}{2.26098 - 1}\right] = \$2819.50$$

This is annual amount required to retire the loan. The dollar–time diagram is Figure 6.7.

Table 6.4 is a detailed illustration of Example 6.8 showing that paying \$2819.55 annually will retire the loan in 10 years. Note that the amount of interest paid is slowly reduced as the outstanding principal is reduced. Over the 10-year period, \$9695.99 interest was paid on the \$18,500 loan.

Industrial loans may be compounded monthly, quarterly, semiannually, or annually. Personal loans, however, on home mortgages and automobile loans are compounded monthly. Example 6.9 demonstrates the effect of monthly compounding on Example 6.8.

Example 6.9

Suppose a person wants to obtain a home equity loan of \$18,500 for remodeling. The interest rate is 8.5% compounded monthly and it is agreed to retire the loan in 5 years. How much will the monthly payments be?

$$A = P\left[\frac{(i)(1+i)^n}{(1+(i)^n) - 1}\right] = (\$18{,}500)\left[\frac{(0.085/12)(1+(0.085))^{60}}{(1+(0.085))^{60} - 1}\right]$$

$$= (\$18{,}500)\left[\frac{(0.00708)(1.00708)^{60}}{(1.00708)^{60} - 1}\right] = (\$18{,}500)\left[\frac{(0.00708)(1.5270)}{(0.5270)}\right]$$

$$= \$379.52$$

TABLE 6.4 Detailed Illustration of Example 6.8

Year	Amount owed at beginning of period	Interest due for period at 8.5%	Amount paid on principal	Balance of principal owed after payment
0				$18,500.00
1	$18,500.00	$1,572.50	$1,247.05	17,252.95
2	17,252.95	1,466.50	1,353.05	15,899.50
3	15,899.50	1,351.49	1,468.06	14,431.44
4	14,431.44	1,226.67	1,592.86	12,838.56
5	12,838.56	1,091.28	1,728.27	11,110.29
6	11,110.29	944.37	1,857.18	9,235.11
7	9,235.11	784.98	2,034.57	7,200.54
8	7,200.54	612.05	2,207.50	4,993.04
9	4,993.04	424.41	2,395.14	2,597.90
10	2,597.90	221.74	2,597.90	0
Total		9,695.90	18,500.00	

34.4% of the total payment was interest.

Note that the monthly interest rate, 8.5%, is divided by 12 months and that in turn is raised to the 60th power to account for the 60 months to retire the loan. Therefore, the person must pay $379.52 every month for 5 years. The total payment is $22,771.20 over the 5-year period and $4,271.20 is the interest.

6.6 EFFECTIVE INTEREST RATES

Nominal interest rates as noted previously are on an annual basis but effective interest rates may be for any period. When one borrows or invests money, although a nominal interest is stated, an effective rate must be given according to government regulations. The term APY is the effective rate for money loaned and yield is the corresponding effective term for money invested [1].

The effective interest rate is calculated from the following expression

$$i_{\text{eff}} = \left[1 + \left(\frac{i}{m}\right)\right]^{(m)(1)} - 1 \tag{6.11}$$

Note that the time period for calculating the effective interest rate is 1 year.

Example 6.10

A person is quoted an 8.33% interest rate on a 4-year loan and the interest is compounded monthly. What is the effective interest rate on this purchase?

$$i_{\text{eff}} = \left[1 + \left(\frac{0.0833}{12}\right)\right]^{(12)(1)} - 1 = (1.006941)^{(12)} - 1 = 1.0865 - 1$$

$$= 0.0865, \quad \text{or} \quad 8.65\%$$

6.7 CHANGING INTEREST RATES

Thus far in this chapter, it was assumed that the interest rates were constant over the period. Interest rates may change appreciably over time depending on economic conditions [1]. To illustrate how the changes are handled, let's consider that a sum of $5000 was placed in a certificate of deposit for three 3-year periods with the corresponding interest rates of 5.5, 4.7, and 4.0%, respectively.

If we let C_1 be the balance at the end of the first 3-year period at 5.5%, then

$$C_1 = (\$5000)(F/P\ 5.5\%,3) = (\$5000)(1 + 0.0550)^3 = (\$5000)(1.1742)$$

$$= \$5871.00$$

For the second period, C_1 is the amount invested or P. C_2, the balance at the end of the second 3-year period at 4.7% interest, becomes

$$C_2 = (\$5871.00)(F/P\ 4.7\%,3) = (\$5871.00)(1 + 0.0470)^3$$

$$= (\$5871.00)(1.1477) = \$6738.33$$

C_2 becomes the P for the third 3-year period; therefore, the balance at the end of the third 3-year period becomes

$$C_3 = (\$6738.33)(F/P\ 4.0\%,3) = (\$6738.33)(1 + 0.040)^3$$

$$= (\$6738.33)(1.1249) = \$7579.95$$

The \$5000 deposited in the CD grew to \$7579.95 over the 9 years at the specified interest rates.

6.8 SUMMARY OF COMPOUND INTEREST FACTORS

Table 6.5 is a summary of the discrete compound interest factors developed in the previous sections. The equations for each factor and the short-hand destinations are given. It is useful to have these equations summarized in one place for use in subsequent chapters.

TABLE 6.5 Summary of Discrete Compound Interest Factors

Factor	Find	Given	Discrete payments Discrete compounding	
Single payment				
Compound amount	F	P	$F = P(1+i)^n$	$P(F/P\ i,n)$
Present worth	P	F	$P = F\frac{1}{(1+i)^n}$	$F(P/F\ i,n)$
Uniform series				
Compound amount	F	A	$F = A\left[\frac{(1+i)^n-1}{i}\right]$	$A(F/A\ i,n)$
Sinking fund	A	F	$A = F\left[\frac{i}{(1+i)^n-1}\right]$	$F(A/F\ i,n)$
Present worth	P	A	$P = A\left[\frac{(1+i)^n-1}{i(1+i)^n}\right]$	$A(F/A\ i,n)$
Capital recovery	A	P	$A = P\left[\frac{i(1+i)^n}{(1+i)^n-1}\right]$	$P(A/P\ i,n)$

Source: Ref. 1.

6.9 CONTINUOUS INTEREST

In some organizations, namely, petroleum and chemical companies, transactions occur hourly or daily and the receipts from sales and/or services are invested immediately. The interest on this cash flow is continuously compounded [1]. These same companies use continuous compounding when evaluating investments in proposed ventures, since the assumption is that cash flows continuously [2].

In continuous compounding, the year is divided into an infinite number of periods [2]. The limit of the interest term is

$$\lim\left(1+\frac{r}{m}\right)^{mn} \text{ as } n \text{ approaches infinity is } e^{rn}$$

where

n = number of years
m = number of interest periods per year
r = nominal annual interest rate
e = base for the Naperian logarithms

In Example 6.2, as the compounding period decreases, the compounding factor increases.

$$F = P[e^{rn}] = (\$2500)[e^{(0.10)(5)}] = (\$2500)(1.648721) = \$4121.80$$

Table 6.6 illustrates the effect of increasing the number of compounding periods and, although the difference in the compounding factor is small, the ultimate is continuous compounding.

6.9.1 Continuous Interest Factors

6.9.1.1 Single-Payment Compound Amount Factor

Under continuous compounding, Eq. (6.5) becomes

$$F = Pe^{rn} \tag{6.12}$$

TABLE 6.6 Effect of Increasing the Number of Compounding Periods

Interest type	Period	Relationship	If $i = 15\%$	Factor
Simple	Annually	$(1 + in)$	(1.15)	1.150000
Compound	Annually	$(1 + i)^1$	(1.15)	1.150000
	Semiannually	$(1 + i/2)^2$	$(1.075)^2$	1.155625
	Quarterly	$(1 + i/4)^4$	$(1.0375)^4$	1.158650
	Monthly	$(1 + i/12)^{12}$	$(1.0125)^{12}$	1.160755
	Daily	$(1 + i/365)^{365}$	$(1.000410958)^{365}$	1.161798
Continuous	—	e^i	$e^{0.15}$	1.161834

In shorthand format

$$F = P(F/P\ r,n)^{\infty}$$

The superscript ∞ denotes that continuous compounding is being used. Example 6.11 illustrates how this equation is used [1].

Example 6.11

If $10,000 is deposited in a savings account at 5% interest compounded continuously, what is the amount at the end of 7 years?

$$F = P(F/P\ r,n)^{\infty} = Pe^{rn} = (\$10{,}000)e^{(0.05)(7)} = (\$10{,}000)(1.419067)$$
$$= \$14{,}190.67$$

6.9.1.2 Single-Payment Present Worth Factor

This factor is the inverse of the single-payment compound amount factor, or

$$P = F\left(\frac{1}{e^{rn}}\right) \tag{6.13}$$

The shorthand form is

$$P = F(P/F\ r,n)^{\infty} = F\left(\frac{1}{e^{rn}}\right) \tag{6.14}$$

6.9.1.3 Uniform Series Compound Amount Factor

When a single payment in a uniform series is considered individually, the future amount is

$$F = A[1 + e^{r} + e^{2r} + \cdots + e^{r(n-1)}] \tag{6.15}$$

The geometric progression with the common ratio e^{r} reduces to the sum of n terms as

$$\frac{e^{rn} - 1}{e^{r} - 1} \tag{6.16}$$

Therefore,

$$F = A\left[\frac{e^{rn} - 1}{e^{r} - 1}\right] = A(F/A\ r,n)^{\infty} \tag{6.17}$$

The term in brackets is the series compound amount factor [1]. Example 6.12 illustrates how this equation is used.

Example 6.12

If \$1000 is deposited every year at the end of the year in an account earning 4.5% interest compounded continuously, how much will be in the account at the end of 5 years?

$$F = A(F/A\ r,n)^{\infty} = A\left[\frac{e^{rn} - 1}{e^{r} - 1}\right] = (\$1000)\left[\frac{e^{(0.045)(5)} - 1}{e^{(0.045)} - 1}\right]$$

$$= (\$1000)\left[\frac{1.252323 - 1}{1.046027 - 1}\right] = (\$1000)(5.48206) = \$5482.06$$

If an amount is deposited quarterly, the method of calculating the amount at the end of a period is shown in Example 6.13.

Example 6.13

Suppose that in Example 6.12, \$1000 is deposited in an account quarterly at 4.5% interest compounded continuously. What is the amount in the account at the end of 5 years?

$$F = A(F/A\ r,n)^{\infty} = (\$1000)[F/A(4.5/4);(5)(4)]^{\infty} = A\left[\frac{e^{rn} - 1}{e^{r} - 1}\right]$$

$$= (\$1000)\left[\frac{e^{(0.045/4)(5)(4)} - 1}{e^{(0.045/4)} - 1}\right] = (\$1000)\left[\frac{1.252323 - 1}{1.011314 - 1}\right]$$

$$= (\$1000)(22.3018) = \$22{,}301.80$$

6.9.1.4 Uniform Series Sinking Fund Factor

As in the case of the discrete interest equations, this factor is the inverse of the uniform series compound amount factor, or

$$A = F\left[\frac{e^{r} - 1}{e^{rn} - 1}\right] = F(A/F\ r,n)^{\infty} \tag{6.18}$$

6.9.1.5 Uniform Series Present Worth Factor

A combination of the single-payment compound present worth factor, $(P/A\ r,n)^{\infty}$, and the uniform series compound amount factor, $(F/A\ r,n)^{\infty}$, will result in

$$(P/A\ r,n)^{\infty} = \frac{e^{rn} - 1}{(e^{rn})(e^{r} - 1)} \tag{6.19}$$

or

$$P = A(P/Ar,n)^{\infty} = A\left[\frac{e^{rn} - 1}{(e^{rn})(e^{r} - 1)}\right] \tag{6.20}$$

6.9.1.6 Uniform Series Capital Recovery Factor

This is the inverse of the uniform series present worth factor.

$$(A/P\, r,n)^{\infty} = \frac{(e^{rn})(e^{r} - 1)}{e^{rn} - 1} \tag{6.21}$$

and

$$A = P(A/P\, r,n)^{\infty} = P\frac{(e^{rn})(e^{r} - 1)}{e^{rn} - 1} \tag{6.22}$$

Example 6.14 illustrates the use of Eq. (6.22) [1].

Example 6.14

If \$1000 was deposited in an account now at 4% interest compounded continuously and it is desired to withdraw 5 equal payments so that the fund will be depleted in 5 years, what would be the amount of each equal withdrawal?

$$A = P(A/P\, r,n)^{\infty} = P\left[\frac{(e^{rn})(e^{r} - 1)}{e^{rn} - 1}\right] = (\$1000)\left[\frac{e^{(0.04)(5)}(e^{0.04} - 1)}{e^{(0.04)(5)} - 1}\right]$$

$$= (\$1000)\left[\frac{(e^{(0.2)})(e^{0.04} - 1)}{e^{(0.2)} - 1}\right] = (\$1000)\left[\frac{(1.221403)(1.040811 - 1)}{1.221403 - 1}\right]$$

$$= (\$1000)\left[\frac{0.049847}{0.221403}\right] = (\$1000)(0.225141) = \$225.14$$

6.10 EFFECTIVE INTEREST WITH CONTINUOUS COMPOUNDING

The effective interest rate using continuous computing is

$$i_{\mathrm{eff}} = e^{r} - 1 = (F/P\, r,1)^{\infty} - 1 \tag{6.23}$$

In this case, e^{r} replaces the $(1 + r/m)^{m}$ in the discrete interest case [1,2]. A 15% nominal interest rate compounded continuously is 16.1834% compared with 16.1798% for daily compounding. Note that each result has been carried out to the same number of significant figures.

6.11 COMPARISON OF ALTERNATIVES

In an economic evaluation, the cash flow of alternative ventures involves the comparison of the income and disbursement of funds. These funds will be different between alternatives for each period considered. So, too, the initial capital investment for each alternative will probably be different. It is therefore necessary to compare and evaluate these alternatives on a common basis, e.g., the present time [1,2]. The may be done in the following ways:

1. By discounting all cash flows back to their present value and comparing the present values of the alternatives.
2. By compounding all cash flows forward in time to their future value and again comparing the future values of the alternatives.
3. By converting all cash flows into equivalent uniform annual costs and annual worths and comparing the equivalent for each alternative.
4. By determining the rate of return or economic value added for each alternative and comparing the alternatives. The techniques for doing this will be considered at length in the chapters on Profitability and Choice Between Alternatives.

6.12 CAPITALIZED COST

The capitalized cost method is a present analysis in which the economic life of an asset or venture is considered indefinitely long [1–3]. Dams, canals, railroad beds, tunnels, etc., are all installations that could be considered to have indefinitely long lives. The continuous compound interest factors in Table 6.7

TABLE 6.7 Summary of Continuous Compound Interest Factors

Factor	Find	Given	Continuous compounding	
Single payment				
Compound amount	F	P	$F = P(e^{rn})$	$P(F/P\ r,n)^{\infty}$
Present worth	P	F	$P = F(e^{-rn})$	$F(P/F\ r,n)^{\infty}$
Uniform series				
Compound amount	F	A	$F = A\left[\frac{e^{rn}-1}{e^{r}-1}\right]$	$F(F/A\ r,n)^{\infty}$
Sinking fund	A	F	$A = F\left[\frac{e^{r}-1}{e^{rn}-1}\right]$	$F(A/F\ r,n)_{\infty}$
Present worth	P	A	$P = A\left[\frac{e^{rn}-1}{e^{rn}(e^{r}-1)}\right]$	$A(P/A\ r,n)^{\infty}$
Capital recovery	A	P	$A = P\left[\frac{e^{rn}(e^{r}-1)}{e^{rn}-1}\right]$	$A(P/A\ r,n)^{\infty}$

Source: Ref. 1.

may be used, since n is considered to be infinity. In the chemical process industries, this method is not used.

REFERENCES

1. JR Couper, WH Rader. Applied Finance and Economic Analysis for Scientists and Engineers. New York: Van Nostrand Reinhold, 1986.
2. EL Grant, WG Ireson, RS Leavenworth. Principles of Engineering Economy. 6th ed. New York: Wiley, 1976.
3. KK Humphreys, ed. Jelen's Cost and Optimization Engineering. 3rd ed. New York: McGraw-Hill, 1991.

PROBLEMS

6.1 If Ted invests \$6000 at 10% interest compounded annually, how much will the investment be worth in 10 years? What total interest will be earned?

6.2 If a person borrows \$6000 at 10% interest compounded annually and pays it back with equal annual payments in 10 years, how much will each payment be? How much interest will be paid in the 10-year period?

6.3 Joan borrows \$6000 at 8% interest compounded monthly and pays it back in equal monthly payments in 10 years. How much will each payment be? How much interest will she have paid in the 10-year period?

6.4 If Joe wanted to have \$10,000 in the bank in 7 years and certificates of deposit were paying 5% interest compounded monthly, how much should be invested in the CD?

6.5 If \$150 per month were put into a credit union that paid 4.5% interest, how much would be in the account at the end of 12 years?

6.6 What is the effective interest rate in Problem 6.3?

6.7 What is the effective interest rate in Problem 6.4?

6.8 A couple takes out a \$75,000 mortgage on a house at 8% interest compounded monthly and agrees to pay it off in 20 years with monthly payments. How much are the monthly payments? How much will be left to pay off after the 100th payment?

6.9 If \$1616.50 is paid back at the end of 6 years for \$1000 borrowed 6 years ago, what interest compounded annually was paid?

6.10 A person is considering depositing an amount in a "savings" account. Various interest rates are listed below. Which is the best for the depositor?

a. 5% compounded annually
b. 4.5% compounded semiannually
c. 4.2% compounded quarterly
d. 4.4% compounded continuously

6.11 How many years will it take to double a sum of money at 5% interest compounded annually?

6.12 Skinflint Loan Company offers the following personal loan plan called the "1% plan." For a 10-month repayment period, this operates as follows: The company adds 10% (i.e., 1% for each month) to the amount borrowed; the borrower pays back one-tenth of this total at the end of each month for 10 months. Thus, for each $100 borrowed, the monthly payment is $11. Find the nominal and effective interest rates per annum.

6.13 A revolving charge account is opened at a local department store that charges 1.5% on the unpaid balance. What is the effective annual interest rate if charges are monthly?

6.14 What is the uniform annual amount that is equivalent to the following series of cash flows at the end of each year if the interest rate is

a. 7% compounded annually
b. 7% compounded continuously

End of year	Amount
0	$5000
1	1000
2	1200
3	1400
4	1600
5	1800
6	2000
7	2200
8	2400

6.15 A man places $5000 in a CD account for 3 years paying 4.5% compounded monthly. At the end of 3 years he withdraws one-third of the money and leaves the remainder in a 3-year CD at 5%. How much is in the CD account at the end of the sixth year?

6.16 If $15,000 were borrowed at 8% interest compounded continuously for 7 years, what would the quarterly payments be?

6.17 Dorothy invests $3000 in a money market account at 4.75% interest compounded monthly. At the end of the twelfth month she deposits $100 into the account and continues to do so every month for 3 years, making 37 such monthly deposits. At the end of 5 years from time 0, assuming a constant interest rate, how much money is in the account?

6.18 A father wishes to establish a fund for his son's college education. He wishes the fund to provide $15,000 on the son's eighteenth, nineteenth, twentieth, and twenty-first birthdays. He will make the first deposit when the son is 6 months old and will make semiannual deposits thereafter until the boy is 17 years and 6 months. If 5% compounded semiannually can be earned on the fund, what should the semiannual deposit be? *Note*: there are 35 deposits and realize that the problem has two parts. (Adapted from Ref. 2.)

6.19 A debt of unknown amount is entered into at time 0. The debtor has agreed to pay $5000 at the end of year 1, $4000 at the end of year 2, $3000 at the end of year 3, $2000 at the end of year 4, and $1000 at the end of year 5. The debt will be fully settled at that time. At 8% interest compounded annually, how much was borrowed at time 0?

6.20 Find the present worth of a uniform cash flow of $150,000 per year over a 5-year period at 15% continuous interest.

6.21 What is the present worth of a cash flow that occurs during the fifth year (4 to 5) of a project? The normal discounting rate of interest is 12% compounded continuously.

6.22 What is the present worth at start-up time (time 0) for $100M of land purchased or allocated to a project 3 years before the plant start-up assuming 16% continuous interest?

6.23 The present worth is desired for construction costs of $10MM occurring uniformly over 2 years prior to start-up (year − 2) to start-up (year 0). Interest is compounded continuously at 13%.

6.24 Determine the present worth of maintenance charges that increase continuously from 0 to $100,000 per year over a 10-year period. The interest rate is 15% compounded continuously.