C H A P T E R

1

Introduction to Newtonian Mechanics

1.1 INTRODUCTION

Mechanics is one of the oldest and most familiar branches of physics. It deals with bodies at rest and in motion and the conditions of rest and motion when bodies are under the influence of internal and external forces. The laws of mechanics apply to a whole range of objects, from microscopic to macroscopic, such as the motion of electrons in atoms and that of planets in space or even to the galaxies in distant parts of the universe.

Mechanics does not explain why bodies move; it simply shows how a body will move in a given situation and how to describe such motion. The study of mechanics may be divided into two parts: kinematics and dynamics. *Kinematics* is concerned with a purely geometrical description of the motion (or trajectories) of objects, disregarding the forces producing the motion. It deals with concepts and the interrelation between position, velocity, acceleration, and time. *Dynamics* is concerned with the forces that produce changes in motion or changes in other properties, such as the shape and size of objects. This leads us to the concepts of force and mass and the laws that govern the motion of objects. *Statics* deals with bodies at rest under the influence of external forces.

In antiquity significant gains were made in the theory of mechanics during Aristotle's time; however, it was not until the seventeenth century that the science of mechanics was truly founded by Galileo, Huygens, and Newton. They showed that objects move according to certain rules, and these rules were stated in the form of laws of motion. Essentially classical or Newtonian mechanics is the study of the consequences of the laws of motion as formulated by Newton in his *Philosophiae Naturalis Principia Mathematica* (the *Principia*) published in 1686.

Although Newton's laws provide a direct approach to the subject of classical mechanics, there are a number of other ways of formulating the principles of classical mechanics. Among these, the two most significant approaches are the formulations of Lagrange and Hamilton.

These two approaches take *energy* rather than force as the fundamental concept. In more than half of this text, we will use the classical approach of Newton, while in the later part of the text we will introduce Lagrange and Hamilton formulations.

Until the beginning of the twentieth century, Newton's laws were completely applicable to all well-known situations. The difficulties arose when these laws were applied to certain definite situations: (a) to very fast moving objects (objects moving with speeds approaching the speed of light) and (b) to objects of microscopic size such as electrons in atoms. These difficulties led to modifications in the laws of Newtonian mechanics: (a) to the formulation of the *special theory of relativity* for objects moving with high speeds, and (b) to the formulation of *quantum mechanics* for objects of microscopic size. The failure of classical mechanics in these situations is the result of inadequacies in classical concepts of space and time as discussed briefly in Chapter 16, Special Theory of Relativity.

Before we start an in-depth study of mechanics, we devote this chapter to summarizing briefly a few essential concepts of interest from introductory mechanics. We especially emphasize the importance of the role of Newton's laws of motion.

1.2 UNITS AND DIMENSIONS

Measurements in physics involve such quantities as velocity, force, energy, temperature, electric current, magnetic field, and many others. The most surprising aspect is that all these quantities can be expressed in terms of a few basic quantities, such as length L, mass M, and time T. These three quantities are called *fundamental* or *basic quantities* (*base units*); all others that are expressed in terms of these are called *derived quantities.*

Three Basic Standards: Length, Mass, and Time

Three different sets of units are in use. The most prevalent is that in which length is measured in *meters*, mass in *kilograms*, and time in *seconds*, hence the name *MKS system* (or *metric system*). As we will see, in practice there are five different quantities that are used as base units.

Standard of length: The meter. The meter has been defined as the distance between the two marks on the ends of a platinum-iridium alloy metal bar kept in a temperature-controlled vault at the International Bureau of Weights and Measures in Sèvres, near Paris, France. In 1960, by international agreement, the General Conference on Weights and Measures changed the standard of length to an atomic constant by the following procedure. A glass tube is filled with krypton gas in which an electrical discharge is maintained. The standard *meter* is defined to be equal to exactly 1,650,763.73 wavelengths of orange-red light emitted in a vacuum from krypton-86 atoms. To improve the accuracy still further, a meter was redefined in 1983 as equal to a distance traveled by light in vacuum in a time interval of 1/299,792,458 of a second.

Standard of time: The second. In the past, the spinning motion of Earth about its axis, as well as its orbital motion about the Sun, have been used to define a second. Thus, a second is defined to be 1/86,400 of a mean solar day. In October 1967, the time standard was redefined in terms of an atomic clock, which makes use of the periodic atomic vibrations of certain atoms. According to the cesium clock, a *second* is defined to be exactly equal to the time interval of 9,192,631,770 vibrations of radiation from cesium-133. This method has an accuracy of 1 part in 10^{11}. It is possible that two cesium clocks running over a period of 5000 years will differ by only 1 second.

Standard of mass: The kilogram. A platinum-iridium cylinder is carefully stored in a repository at the International Bureau of Weights and Measures. The mass of the cylinder is defined to be exactly equal to a *kilogram*. This is the only base unit still defined by an artifact. The basic aim of scientists has been to define the three basic standards in such a way that they are accurately and easily reproducible in any laboratory.

Different Systems of Units

Besides MKS system, there are two others, all using five base units, which are briefly described below.

The CGS or Gaussian system. In this system the unit of length is the *centimeter* (= 10^{-2} m), the unit of mass is the *gram* (= 10^{-3} kg), and the unit of time is the *second*.

The British system. This is used in the United States and may be referred to as U.S. engineering system. In this system the unit of length is the *foot* and the unit of time is the *second*. This system does not use mass as a basic unit; instead, *force* is used, the unit of which is the *pound* (lb). The unit of mass derived from the pound is called the *slug* (= 32.17 lb mass). The unit of temperature in the British system is the *degree Fahrenheit*.

The MKS or metric system. In this system the unit of length is the *meter* (m), the unit of mass is the *kilogram* (kg), and the unit of time is the *second* (sec). These are the most commonly used units in the world. The other two base units are temperature in kelvins (K) and charge in coulombs (coul).

Five of the most commonly used base units in the different systems are listed here.

Units	MKS	CGS	USA
Length	$L := 1 \cdot m$	$L := 1 \cdot cm$	$L := 1 \cdot ft$
Mass	$M := 1 \cdot kg$	$M := 1 \cdot gm$	$L := 1 \cdot lb$
Time	$T := 1 \cdot sec$	$T := 1 \cdot sec$	$T := 1 \cdot sec$
Temperature	$R := 1 \cdot K$	$R := 1 \cdot K$	$R := 1 \cdot K$
Charge	$Q := 1 \cdot coul$	$Q := 1 \cdot coul$	$Q := 1 \cdot coul$

International System of Units (SI). The International System of Units, abbreviated SI after the French *Système international d'unités,* is the modern version of the metric system established by international agreement. For convenience it uses 7 base units: Five of these are the same as MKS already listed and the other two are:

Amount of substance	mole	$1 \cdot$ mol
Luminous intensity	candela	$1 \cdot$ cd

The SI also uses two supplementary units:

Plane angle	radian	$1 \cdot$ rad
Solid angle	steradian	$1 \cdot$ sr

Dimensions

Most physical quantities may be expressed in terms of length L, mass M, and time T, where L, M, and T are called dimensions. A quantity expressed as $L^aM^bT^c$ means that its length dimension is raised to the power a, its mass dimension is raised to the power b, and its time dimension is raised to the power c. Thus the dimensions of volume are L^3, that of acceleration are LT^{-2}, and that of force are MLT^{-2}.

To add or substract two quantities in physics they must have the same dimensions. Similarly, no matter what system of units is used, all mathematical relations and equations must be dimensionally correct. That is, the quantities on both sides of the equations must have the same dimensions. For example, in the equation $x = v_0 + \frac{1}{2}at^2$, x has dimensions of L, v_0t has dimensions of $(L/T)T = L$, and $\frac{1}{2}\,at^2$ has dimensions of $\frac{1}{2}(L/T^2)(T^2) = L$. Thus dimensional analysis may be used to (1) check the correctness of the form of the equation, that is, every term in the equation must have the same dimensions, (2) to check an answer computed from an equation for plausibility in a given situation, and (3) to arrive at a formula if we know the dependence of a certain quantity on other physical quantities.

Example 1.1

The magnitude of the centripetal force Fc acting on an object is a function of mass M of the object, its velocity v, and the radius r of the circular path. By the method of dimensional analysis, find an expression for the centripetal force.

Solution

Since Fc is a function of M, v, and r, the values of a, b, and c are calculated in the expression for Fc.

$$Fc = M^a \cdot v^b \cdot r^c \qquad \text{(i)}$$

$$Fc := 1 \cdot kg \cdot \frac{m}{sec^2} \qquad M := 1 \cdot kg \qquad v := 1 \cdot \frac{m}{sec} \qquad r := 1 \cdot m$$

In terms of units, expression (i) takes the form (ii).

$$kg \cdot \frac{m}{sec^2} = kg^a \cdot \left(\frac{m}{sec}\right)^b \cdot m^c \qquad \text{(ii)}$$

We assume the values of a, b, and c to be (guess values)

Guess

$a := 1 \qquad b := 1 \qquad c := 1$

Comparing the values of a, b, and c on both sides of Eq. (ii), we get Eqs. (iii).

Given

$$a = 1 \qquad b + c = 1 \qquad b - 2 = 0 \qquad \text{(iii)}$$

Let S represent the solution giving the values of a, b, and c that satisfy Eqs. (i) and (ii). The results are: a =1, b = 2 and c = -1.

$S := \text{Find}(a, b, c)$

$$S = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

Thus the proper equation for force Fc and its units are

$$Fc = \frac{Mv^2}{r} \qquad Fc = 1 \cdot kg \cdot m \cdot sec^{-2} \qquad \text{(iv)}$$

EXERCISE 1.1 The angular velocity ω of a simple pendulum is a function of its length L and acceleration due to gravity g. Find an expression for the angular velocity ω and the time period T of the pendulum by the method of dimensional analysis.

1.3 NEWTON'S LAWS AND INERTIAL SYSTEMS

Newton's laws may be stated in a brief and concise form as below:

> ***Newton's First Law.*** *Every object continues in its state of rest or uniform motion in a straight line unless a net external force acts on it to change that state.*
>
> ***Newton's Second Law.*** *The rate of change of momentum of an object is directly proportional to the force applied and takes place in the direction of the force.*
>
> ***Newton's Third Law.*** *To every action there is always an equal and opposite reaction; that is, whenever a body exerts a certain force on a second body, the second body exerts an equal and opposite force on the first.*

These statements do look simple; but that is deceptive. Newton's laws are the results of a combination of definitions, experimental observations from nature, and many intuitive concepts. We cannot do justice to these concepts in a short space here, but we will try to expand our thinking horizon by discussing these statements in some detail.

The motion of objects in our immediate surroundings is complicated by ever present frictional and gravitational forces. Let us consider an isolated object that is moving with a constant (or uniform) velocity in space. Describing it as an isolated object implies that it is far away from any surrounding objects so that it does not interact with them; hence no net force (gravitational or otherwise) acts on it. To describe the motion of the object we must draw a coordinate system with respect to which the object moves with uniform velocity. Such a coordinate system is called an *inertial system.* The essence of Newton's first law is that it is always possible to find a coordinate system with respect to which an isolated body moves with uniform velocity; that is, *Newton's first law asserts the existence of inertial systems.*

Newton's second law deals with such matters as: What happens when there is an interaction between objects? How do you represent interaction? And still further, what is inertia and how do we measure this property of an object? As we know, *inertia* is a property of a body that determines its resistance to acceleration or change in motion when that body interacts with another body. The quantitative measure of *inertia* is called *mass.*

Consider two bodies that are completely isolated from the surroundings but interact with one another. The interaction between these objects may result from being connected by means of a rubber band or a spring. The interaction results in acceleration of the bodies. Such accelerations may be measured by stretching the bodies apart by the same amount and then measuring the resultant accelerations. All possible measurements show that the accelerations of these two bodies are always in opposite directions and that the ratio of the accelerations are inversely proportional to the masses. That is,

$$\frac{a_A}{a_B} = -\frac{m_B}{m_A}$$

or

$$m_A a_A = -m_B a_B \tag{1.1}$$

Thus the effect of interaction is that the product of mass and acceleration is constant and denotes the *change in motion.* This product is called *force* and it represents interaction. Thus we may say that the force F_A acting on A due to interaction with B is

$$F_A = m_A a_A \tag{1.2}$$

while the force F_B acting on B due to interaction with A is

$$F_B = m_B a_B \tag{1.3}$$

Thus, in general, using vector notation we may write

$$\mathbf{F} = m\mathbf{a} \tag{1.4}$$

This equation is the definition of force when acting on a constant mass and holds good only in inertial systems. It is important to keep in mind that the force $\mathbf{F}$ arises because of an interaction or simply stands for an interaction. No acceleration could ever be produced without an interaction.

Let us now proceed to obtain the definition of force starting directly with the statement of Newton's second law given previously. Suppose an object of mass m is moving with velocity $\mathbf{v}$ so that the linear momentum $\mathbf{p}$ is defined as

$$\mathbf{p} = m\mathbf{v} \tag{1.5}$$

According to Newton's second law, the rate of change of momentum is defined as force **F**; that is,

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \tag{1.6}$$

This equation takes a much simpler form if mass m remains constant at all speeds. If $\mathbf{v}$ is very small as compared to the speed of light c ($= 3 \times 10^8$ m/s), the variation in mass m is negligible. Hence, we may write

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) = m\frac{d\mathbf{v}}{dt} = m\mathbf{a} \tag{1.7}$$

That is, force is equal to mass (or inertial mass) times acceleration provided m is constant. This is the same as Eq. (1.4). It should be clear that Newton's first law is a special case of the second law, when $\mathbf{F} = 0$.

Let us now say a few words about Newton's third law. According to Newton's third law, forces always exist in pairs. Thus, if two bodies A and B interact with one another, and if there is a force $\mathbf{F}_A$ acting on body A, then there must be a force $\mathbf{F}_B$ acting on body B, so

$$\mathbf{F}_A = -\mathbf{F}_B \tag{1.8}$$

Thus the law implies that forces always exist in pairs (a single force without its partner somewhere else is an impossibility) and that such forces are the result of interactions. We can never have an isolated object having acceleration. An object with acceleration must have a counterpart somewhere else with opposite acceleration that is inversely proportional to mass.

It should be clear that Eq. (1.8) implies that the forces are equal and opposite, but they do not *always* necessarily have the same line of action. These points are elaborated on in Chapter 8.

1.4 INERTIAL AND NONINERTIAL SYSTEMS: NONINERTIAL FORCES

As we mentioned earlier, the first law of motion defines a particular type of reference frame, called the inertial system; that is, the inertial system is one in which Newton's first law holds good. We would like to find a relation between the measurements made by an observer A in an inertial system S and another observer B in a noninertial system S', both observing a common object C that may be moving with acceleration. This situation is shown in Fig. 1.1. S being an inertial system means that observer A is moving with uniform velocity, while system S' being noninertial means that observer B has an acceleration.

Object C of mass M is accelerating. Observer A measures its acceleration to be a_A and observer B measures its acceleration to be a_B. Thus, according to observer A, the force acting on C is

$$F_A = Ma_A \tag{1.9}$$

while according to observer B the force acting on C is

$$F_B = Ma_B \tag{1.10}$$

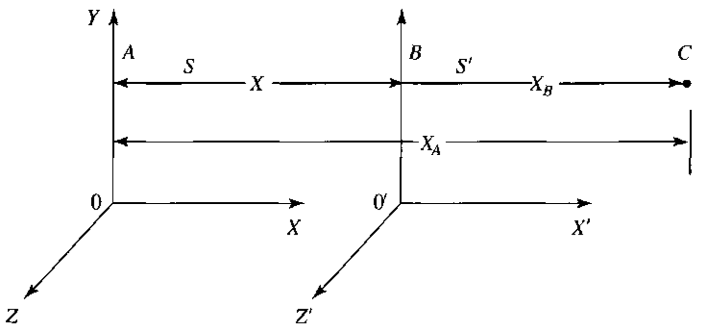


Figure 1.1 Moving object C being observed by an observer A in an inertial system S and another observer B in the noninertial system S'.

F_B would have been equal to F_A if the S' system was also a true inertial system. Let us find the relation between F_A and F_B. According to Fig. 1.1,

$$X_A(t) = X_B(t) + X(t) \tag{1.11}$$

Differentiating twice and rearranging,

$$\ddot{X}_B = \ddot{X}_A - \ddot{X} \tag{1.12}$$

Multiplying both sides by M,

$$M\ddot{X}_B = M\ddot{X}_A - M\ddot{X}$$

or

$$F_B = F_A - M\ddot{X} \tag{1.13}$$

Since observer A in system S is in a true inertial system, we may write

$$F_A = F_{\text{true}} = M\ddot{X}_A \tag{1.14}$$

while the force measured by observer B, who is in a noninertial system S', is not a true force but an apparent force given by

$$F_B = F_{\text{appt}} = M\ddot{X}_B \tag{1.15}$$

Thus we may write Eq. (1.13) as

$$F_{\text{appt}} = F_{\text{true}} - M\ddot{X} \tag{1.16}$$

Thus observer B will not measure a true force unless $\ddot{X} = 0$, in which case B will be moving with a uniform velocity with respect to A; hence S' itself will be a true inertial system. In general, for three-dimensional motion, we may write Eq. (1.16) as

$$\mathbf{F}_{\text{appt}} = \mathbf{F}_{\text{true}} - M\ddot{\mathbf{R}} \tag{1.17}$$

where $\ddot{\mathbf{R}}$ is the acceleration of the noninertial system S' with respect to the inertial system S or with respect to any other inertial system. If $\ddot{\mathbf{R}} = \mathbf{0}$, then $\mathbf{F}_{\text{appt}} = \mathbf{F}_{\text{true}}$, and hence both systems will be inertial. We may write Eq. (1.17) as

$$\mathbf{F}_{\text{appt}} = \mathbf{F}_{\text{true}} + \mathbf{F}_{\text{fict}} \tag{1.18a}$$

where

$$\mathbf{F}_{\text{fict}} = -M\ddot{\mathbf{R}} \tag{1.18b}$$

The last term is called a *noninertial force* or *fictitious force* because it is not a force in the true sense; no interactions are involved. It is simply a product of mass times acceleration.

1.5 SIMPLE APPLICATIONS OF NEWTON'S LAWS

A few simple applications of Newton's laws will be discussed in this section and the next.

Atwood Machine

A system of masses tied with a string and going over a pulley is called an Atwood machine, as shown in Fig. 1.2. We will assume that the pulley is frictionless and hence will not rotate. Mass m_2, being greater than mass m_1, will move downward and m_1 will move upward. The velocity $v = dx/dt$ is taken to be positive upward, while T (which is the same on both sides since the string is massless) is taken to be the tension in each string. Thus the motion of the two masses may be described by the following equations, $a = d^2x/dt^2$ being the acceleration for either mass. Acceleration a is the same on both sides since the string is "stretchless."

$$T - m_1 g = m_1 a \tag{1.19}$$

$$m_2 g - T = m_2 a \tag{1.20}$$

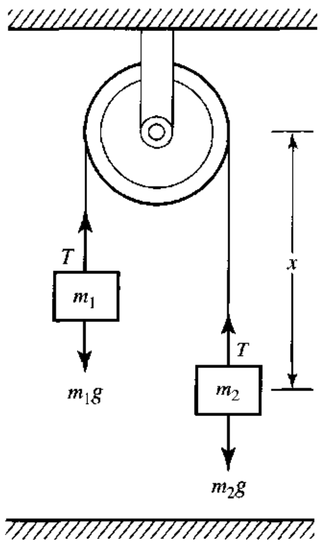


Figure 1.2 Atwood machine.

These equations may be solved to yield

$$a = \frac{m_2 - m_1}{m_1 + m_2} g \tag{1.21}$$

$$T = \frac{2m_1 m_2}{m_1 + m_2} g \tag{1.22}$$

If $m_1 = m_2$, we get $a = 0$, and $T = m_1 g = m_2 g$, which is the case for static equilibrium. On the other hand, if $m_2 \gg m_1$, we get $a \simeq g$ and $T \simeq 2m_1 g$.

Let us consider the case in which the pulley is not stationary but moves upward with an acceleration a, as shown in Fig. 1.3. In such a situation the total length of the string is

$$\ell = \pi R + (y - y_1) + (y - y_2) \tag{1.23}$$

Differentiating, we obtain

$$2\ddot{y} - \ddot{y}_1 - \ddot{y}_2 = 0 \tag{1.24}$$

But $\ddot{y} = a$ is the upward acceleration of the pulley. Hence

$$a = \tfrac{1}{2}(\ddot{y}_1 + \ddot{y}_2) \tag{1.25}$$

These principles can be extended to other situations involving many masses and pulleys.

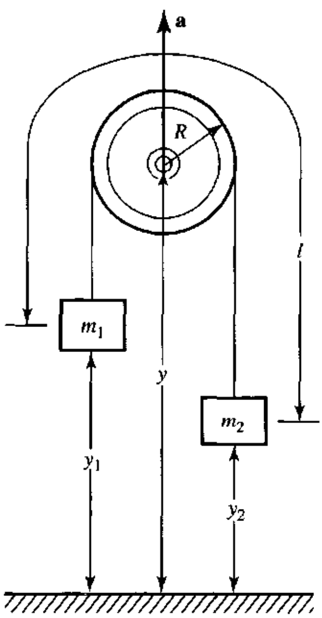


Figure 1.3 Motion of masses when the pulley has an upward acceleration a.

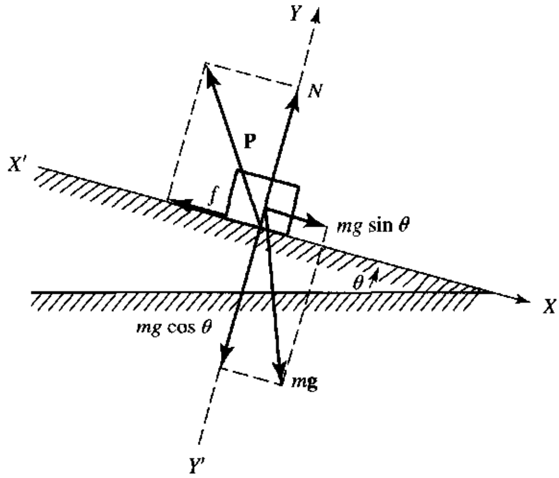


Figure 1.4 Forces acting on a mass m on an inclined plane.

Inclined Plane

Consider a mass m on an inclined plane that makes an angle θ with the horizontal, as shown in Fig. 1.4. The two forces acting on the mass m on the plane are the weight $m\mathbf{g}$ of mass m acting downward and the reaction **P** of the plane acting on mass m as shown. It is the resultant of these two forces that moves the mass up or down the plane, as illustrated for the two cases in Figs. 1.4 and 1.5. **P** can be resolved into two components; component N perpendicular to the surface of the plane is called the *normal reaction* (or normal force), and component f parallel to the surface of the plane is called the *frictional force.*

Let us consider the motion of mass m moving down the inclined plane, as shown in Fig. 1.4. Note that $m\mathbf{g}$ has been resolved into two components $mg \cos \theta$ and $mg \sin \theta$. Thus, for the motion of mass m, we may write

$$\sum F_x = mg \sin \theta - f = ma \tag{1.26}$$

$$\sum F_y = N - mg \cos \theta = 0 \tag{1.27}$$

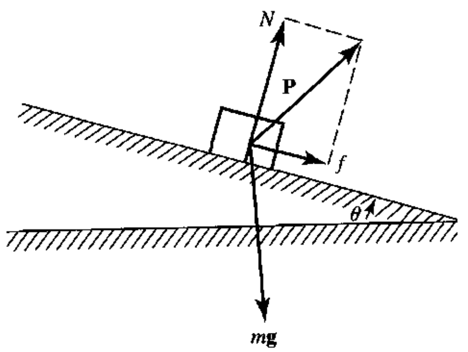


Figure 1.5 Motion of mass m moving upward on an inclined plane.

For a surface with a coefficient of friction μ, the frictional force f is found to be the product of μ and the normal reaction. That is,

$$f = \mu N = \mu mg \cos \theta \tag{1.28}$$

(If the body is at rest $\mu = \mu_s$, where μ_s, is the coefficient of static friction. If the body is in motion, $\mu = \mu_k$, where μ_k is the coefficient of kinetic friction. μ_k is always less than μ_s.) Substituting for f in Eq. (1.26) and solving for a,

$$a = g(\sin \theta - \mu \cos \theta) \tag{1.29}$$

If the mass m has an initial upward velocity along the plane, the direction of f will be opposite, as shown in Fig. 1.5, and the expression for the resulting acceleration (or deceleration) will be

$$a = g(\sin \theta + \mu \cos \theta) \tag{1.30}$$

Suppose mass m is sitting on the horizontal plane and the angle θ of the plane is increased steadily. When the angle reaches a certain value $\theta = \theta_f$, mass m just starts sliding. In this situation, when the motion starts $a = 0$, and we get

$$mg \sin \theta_f - f = 0$$

$$N - mg \cos \theta_f = 0$$

or

$$\frac{f}{N} = \tan \theta_f \tag{1.31}$$

But by definition $f = \mu_s N$, where μ_s is the coefficient of static friction; hence

$$\mu_s = \tan \theta_f \tag{1.32}$$

where θ_f is called the *angle of friction* or the *angle of repose*. If θ is greater than θ_f, the mass will not remain at rest. For the mass to remain at rest,

$$\tan \theta < \tan \theta_f = \mu_s \tag{1.33}$$

The same conclusion may be arrived at by considering Eq. (1.29), according to which the speed of the mass will increase if $a > 0$. This is possible only if

$$(\sin \theta - \mu \cos \theta) > 0$$

That is,

$$\theta > \tan^{-1} \mu \tag{1.34}$$

If $a = 0$, $\theta = \theta_f =$ the angle of friction. If $\theta < \theta_f$, a will be negative, and the particle will not move or will come to rest if already moving.

The Spinning Drum

In a spinning drum or well in an amusement park ride, the riders stand against the wall of the drum. When the drum starts spinning very fast, the bottom of the drum falls down but the riders stay pinned against the wall of the drum. We want to find the minimum angular velocity ω_{min} for which it is safe to remove the bottom.

The situation is as shown in Fig. 1.6, and with N being the unbalanced force the radial equation of motion is

$$N - Ma_r = 0 \tag{1.35}$$

or we could say that the normal reaction N must provide the needed centripetal force F_c:

$$F_c = Ma_r = M\frac{v^2}{R} = MR\omega^2 \tag{1.36}$$

Thus

$$N = Ma_r = MR\omega^2 \tag{1.37}$$

If f is the static frictional force, then

$$f \leqslant \mu_s N = \mu_s MR\omega^2 \tag{1.38}$$

where μ_s is the coefficient of static fricition between the rider and the surface of the drum. For the rider to stay pinned against the wall of the drum, f must be equal to Mg. Substituting this in Eq. (1.38) yields

$$Mg \leqslant \mu_s MR\omega^2 \tag{1.39}$$

or

$$\omega \geqslant \sqrt{\frac{g}{\mu_s R}} \tag{1.40}$$

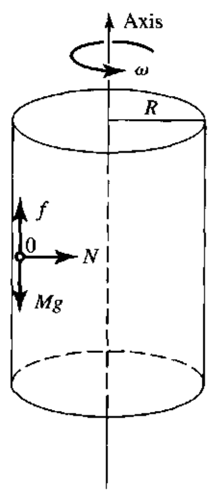


Figure 1.6 Spinning drum.

Thus the minimum safe value of ω is

$$\omega_{\min} = \sqrt{\frac{g}{\mu_s R}} \tag{1.41}$$

Example 1.2

In Figure 1.6 the spinning drum has a radius of 2m and the coefficient of friction is 0.3. Calculate the minimum speed at which it is safe to remove the floor of the spinning drum.

Solution

The different values given are $\mu := 0.3$ $\quad R := 2 \cdot m$ $\quad g := 9.8 \cdot \frac{m}{sec^2}$

From Eq. (1.41), the angular frequency ω of the rotating drum is

$$\omega := \frac{\sqrt{g}}{\sqrt{\mu \cdot R}}$$

$$\omega = 4.041 \cdot sec^{-1} \cdot rad \qquad \omega = 231.558 \cdot sec^{-1} \cdot deg$$

Alternately, the revolution rate and time period are

$$\frac{\omega}{2 \cdot \pi} = 0.643 \cdot sec^{-1} \qquad \frac{2 \cdot \pi}{\omega} = 1.555 \cdot sec$$

Thus if the drum makes 0.643 revolution per second, or it takes 1.555 seconds to make one revolution, it will be safe to remove the drum floor from the bottom.

Should the drum speed be increased or decreased if (a) the coefficient of friction increases or decreases and (b) the radius R is larger or smaller?

EXERCISE 1.2 In the example, suppose we want the drum to rotate at a speed of 2 revolutions per second and still be able to remove the floor safely by: **(a)** changing the radius but keeping μ the same and **(b)** changing μ but keeping the radius the same. What are the values of the radius and μ in the two cases?

Example 1.3

Two blocks of masses m and M are connected by a string and pass over a frictionless pulley. Mass m hangs vertically and mass M moves on an inclined plane that makes an angle θ with the horizontal. If the coefficient of kinetic friction is μ_k, calculate the angle θ for which the blocks move with uniform velocity. Discuss the special case in which $m = M$.

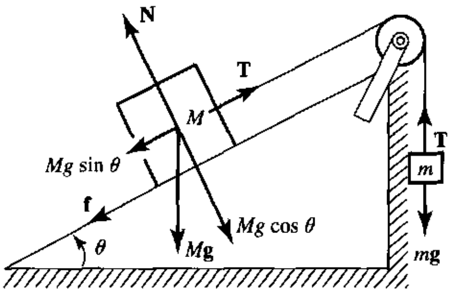


Figure Ex. 1.3

Solution

The situation is as shown in Fig. Ex. 1.3. From the force diagram, if the blocks are in equilibrium, that is, at rest or moving with a uniform velocity,

$$T - mg = 0 \tag{i}$$

$$N - Mg\cos\theta = 0 \tag{ii}$$

$$T - Mg\sin\theta - f = 0 \tag{iii}$$

and, from the definition of the coefficient of friction,

$$f = \mu_k N \tag{iv}$$

Combining Eqs. (ii) and (iv),

$$f = \mu_k Mg\cos\theta \tag{v}$$

Substituting for T from Eq. (i) and for f from Eq. (v) into Eq. (iii)

$$mg - Mg\sin\theta - \mu_k Mg\cos\theta = 0 \tag{vi}$$

First Method:

We can solve the given Eqs. (i), (ii), and (iii) directly. From the force diagram, if the blocks are in equilibrium, at rest, or moving with uniform velocity, the equations after using the definition of the coefficient of friction $f = \mu N$ are as shown.

Solving these equations for the unknowns T, N, and θ yields the values as shown below.

Given

$T - m\cdot g = 0$

$N - M\cdot g\cdot\cos(\theta) = 0$

$T - M\cdot g\cdot\sin(\theta) - \mu\cdot N = 0$

$$\text{Find}(T,N,\theta) \rightarrow \begin{bmatrix} m\cdot g & m\cdot g \\ M\cdot g\cdot\cos\left[2\cdot\text{atan}\left[\frac{\left(M-\sqrt{M^2-m^2+\mu^2\cdot M^2}\right)}{(m+\mu\cdot M)}\right]\right] & M\cdot g\cdot\cos\left[2\cdot\text{atan}\left[\frac{\left(M+\sqrt{M^2-m^2+\mu^2\cdot M^2}\right)}{(m+\mu\cdot M)}\right]\right] \\ 2\cdot\text{atan}\left[\frac{\left(M-\sqrt{M^2-m^2+\mu^2\cdot M^2}\right)}{(m+\mu\cdot M)}\right] & 2\cdot\text{atan}\left[\frac{\left(M+\sqrt{M^2-m^2+\mu^2\cdot M^2}\right)}{(m+\mu\cdot M)}\right] \end{bmatrix}$$

For M = m, the two possible values of θ, given by Sθ, are calculated as shown.

$M := m \qquad \mu := 0.3$

$$S\theta := \begin{bmatrix} 2\cdot\text{atan}\left[\frac{1}{(2\cdot(-m-\mu\cdot M))}\cdot\left(-2\cdot M+2\cdot\sqrt{M^2-m^2+\mu^2\cdot M^2}\right)\right] \\ 2\cdot\text{atan}\left[\frac{1}{(2\cdot(-m-\mu\cdot M))}\cdot\left(-2\cdot M-2\cdot\sqrt{M^2-m^2+\mu^2\cdot M^2}\right)\right] \end{bmatrix}$$

$$S\theta = \begin{pmatrix} 0.988 \\ 1.571 \end{pmatrix}\cdot\text{rad} \qquad S\theta = \begin{pmatrix} 56.602 \\ 90 \end{pmatrix}\cdot\text{deg}$$

Second Method:

Let us now solve Eq. (vi) for the values

$M := m \qquad \mu := 0.3$

Eq. (vi) is written as

$$m\cdot g - M\cdot g\cdot\sin(\theta) - \mu\cdot M\cdot g\cdot\cos(\theta) = 0$$

The two possible values given by S are (solving above equation symbolically for θ)

$$S := \begin{bmatrix} 2\cdot\text{atan}\left[\frac{1}{(2\cdot(-m-\mu\cdot M))}\cdot\left(-2\cdot M+2\cdot\sqrt{M^2-m^2+\mu^2\cdot M^2}\right)\right] \\ 2\cdot\text{atan}\left[\frac{1}{(2\cdot(-m-\mu\cdot M))}\cdot\left(-2\cdot M-2\cdot\sqrt{M^2-m^2+\mu^2\cdot M^2}\right)\right] \end{bmatrix}$$

One value of θ is 56.6 degrees or 0.988 radians while the other value of θ = 90 degrees corresponds to the simple case of two masses hanging vertically from a pulley.

$$S = \begin{pmatrix} 0.988 \\ 1.571 \end{pmatrix}\cdot\text{rad} \qquad S = \begin{pmatrix} 56.602 \\ 90 \end{pmatrix}\cdot\text{deg}$$

EXERCISE 1.3 Find the ratio of the masses M/m so that the two blocks will move with uniform velocity for θ = 45 degrees and μ =0.3.

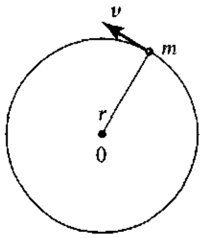


Figure 1.7 Mass m moving in a circle of radius r with a uniform speed v.

1.6 MOTION IN A CIRCLE AND GRAVITATION

Consider a mass m moving in a circle of radius r with a uniform speed v, as shown in Fig. 1.7. The acceleration of mass m is toward the center 0 and is given by

$$a_c = \frac{v^2}{r} \tag{1.42}$$

where a_c is called the *centripetal acceleration* and is produced by a constant force F_c, called the *centripetal force*, and given by

$$F_c = ma_c = m\frac{v^2}{r} \tag{1.43}$$

Note that F_c is not a force in the true sense because it is not produced as the result of interaction between objects. It simply happens to be the product of mass times acceleration.

According to Newton's universal law of gravitation, the gravitational force between mass m at a distance r from the center of Earth of mass M is

$$F_G = G\frac{Mm}{r^2} = mg \tag{1.44}$$

If this point mass is on the surface of Earth, which has a radius R, we may write

$$F = mg_0 = G\frac{Mm}{R^2} \tag{1.45}$$

That is,

$$g_0 = \frac{GM}{R^2} \quad \text{or} \quad G = \frac{g_0R^2}{M} \tag{1.46}$$

Let us assume that m is a satellite or some other object moving with velocity v in a circle of radius r around Earth, as shown in Fig. 1.8. Gravitational force (toward the center of Earth) provides the necessary centripetal force to keep the mass moving in a circular orbit; that is,

$$F_c = F_G$$

$$\frac{mv^2}{r} = G\frac{Mm}{r^2} \tag{1.47}$$

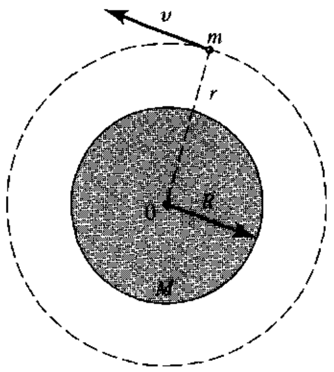


Figure 1.8 Mass m moving in a circle of radius r around Earth of mass M and radius R.

Substituting $v = 2\pi r/T$, where T is the time period of the circular orbit, in Eq. (1.47) and rearranging, we get

$$T^2 = \frac{4\pi^2}{GM} r^3 \tag{1.48}$$

or substituting $GM = gR^2$, we may write Eq. (1.48) as

$$T^2 = \frac{4\pi^2}{gR^2} r^3 \tag{1.49}$$

Equations (1.48) and (1.49) are statements of Kepler's third law.

Horizontal Circular Motion

A small mass m swings in a horizontal circle of radius r at the end of a string of length l_i, which makes an angle θ_i with the vertical as shown in Fig. 1.9. The string is slowly shortened by pulling it through a hole in its support until the final length is l_f and the string is making an angle θ_f with the vertical. Find an expression for l_f in terms of l_i, θ_i, and θ_f.

Suppose at some instant t the length of the string from the support to mass m is l_n, it makes an angle θ_n with the vertical, mass m is at a distance r_n from the axis of rotation, and v_n is the velocity of mass m. Let T_n be the tension in the string as shown. Since mass m is moving in a circle of radius r_n, the horizontal component of the tension in the string must provide the necessary centripetal acceleration; that is,

$$T_n \sin \theta_n = \frac{mv_n^2}{r_n} \tag{1.50}$$

while the vertical component of the tension balances the weight of mass m, that is,

$$T_n \cos \theta_n = mg \tag{1.51}$$

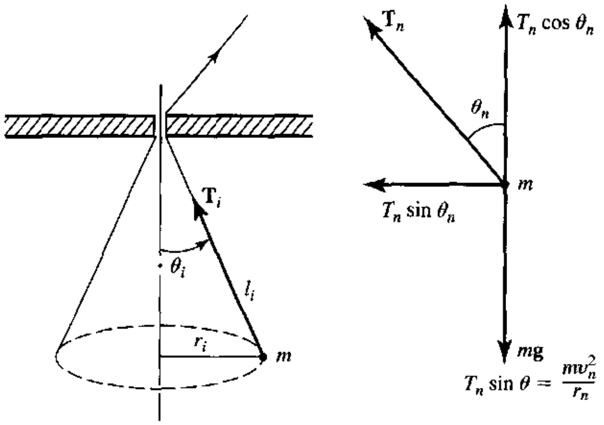


Figure 1.9

Dividing Eq. (1.50) by Eq. (1.51),

$$\tan \theta = \frac{v_n^2}{gr_n} \tag{1.52}$$

which may be written as

$$gr_n^3 \tan \theta_n = v_n^2 r_n^2 \tag{1.53}$$

Since no external torque acts on the system, the angular momentum of the system must be conserved; that is,

$$L_1 = L_2 \quad \text{or} \quad mr_1v_1 = mr_2v_2 \quad \text{or} \quad r_1v_1 = r_2v_2 \tag{1.54}$$

Thus, in general,

$$r_nv_n = \text{constant} \tag{1.55}$$

Combining Eqs. (1.53) and (1.55), we may conclude

$$r_i^3 \tan \theta_i = r_f^3 \tan \theta_f \tag{1.56}$$

But

$$r_n = l_n \sin \theta_n \tag{1.57}$$

Therefore, Eq. (1.56) takes the form

$$(l_i \sin \theta_i)^3 \tan \theta_i = (l_f \sin \theta_f)^3 \tan \theta_f \tag{1.58}$$

which is the required result and may be solved for l_f.

To illustrate the above concepts, we discuss a few examples involving the motion of the planets and the centripetal force.

Example 1.4

(a) Calculate the variation in the value of g with distance from the center of Earth. **(b)** Apply Keplers law to the motion of the Moon around Earth and calculate the distance between them.

Solution

(a) From Eqs. (1.44) and (1.46), gr is the value of g at a distance r from the center of Earth, R0 is radius of Earth, and g0 is the value of g at Earth's surface.

$$gr = \frac{g0 \cdot R0^2}{r^2} = \frac{K}{r^2}$$

From the given values of g0 and R0, the value of K is calculated

$$g0 := 9.806 \qquad R0 := 6.38 \cdot 10^3$$

$$K := g0 \cdot R0^2 \qquad K = 3.991 \cdot 10^8$$

Thus the value of g in terms of r is

$$g = \frac{3.991 \cdot 10^8}{r^2}$$

We calculate g for 60 different values of r and then plot the results as shown in the graph.

$$N := 60 \qquad n := 0..N$$

$$r_n := R0 + R0 \cdot n \qquad g_n := \frac{K}{(r_n)^2}$$

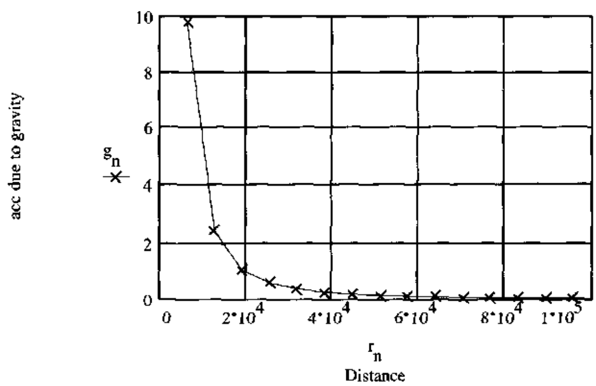
Below are some values of g at different r, together with r_{60} and g_{60}, which correspond to the values near the Moon's surface.

$r_0 = 6.38 \cdot 10^3$ $\qquad g_0 = 9.806$

$r_1 = 1.276 \cdot 10^4$ $\qquad g_1 = 2.451$

$r_{10} = 7.018 \cdot 10^4$ $\qquad g_{10} = 0.081$

$r_{60} = 3.892 \cdot 10^5$ $\qquad g_{60} = 0.003$



Acceleration g versus distance r

(b) Eq. (1.49) for T may be solved for r as shown.

$$T^2 = \left(\frac{4\cdot\pi^2}{g\cdot R^2}\right)\cdot r^3$$

has solution(s)

$$\begin{bmatrix} \frac{1}{4}\cdot 4^{\left(\frac{2}{3}\right)}\cdot T^{\left(\frac{2}{3}\right)}\cdot g^{\left(\frac{1}{3}\right)}\cdot\frac{R^{\left(\frac{2}{3}\right)}}{\pi^{\left(\frac{2}{3}\right)}} \\ \frac{-1}{8}\cdot 4^{\left(\frac{2}{3}\right)}\cdot T^{\left(\frac{2}{3}\right)}\cdot g^{\left(\frac{1}{3}\right)}\cdot\frac{R^{\left(\frac{2}{3}\right)}}{\pi^{\left(\frac{2}{3}\right)}} + \frac{1}{8}\cdot i\cdot\sqrt{3}\cdot 4^{\left(\frac{2}{3}\right)}\cdot T^{\left(\frac{2}{3}\right)}\cdot g^{\left(\frac{1}{3}\right)}\cdot\frac{R^{\left(\frac{2}{3}\right)}}{\pi^{\left(\frac{2}{3}\right)}} \\ \frac{-1}{8}\cdot 4^{\left(\frac{2}{3}\right)}\cdot T^{\left(\frac{2}{3}\right)}\cdot g^{\left(\frac{1}{3}\right)}\cdot\frac{R^{\left(\frac{2}{3}\right)}}{\pi^{\left(\frac{2}{3}\right)}} - \frac{1}{8}\cdot i\cdot\sqrt{3}\cdot 4^{\left(\frac{2}{3}\right)}\cdot T^{\left(\frac{2}{3}\right)}\cdot g^{\left(\frac{1}{3}\right)}\cdot\frac{R^{\left(\frac{2}{3}\right)}}{\pi^{\left(\frac{2}{3}\right)}} \end{bmatrix}$$

T = 27.33 days is the period of revolution of the Moon around Earth and R is the radius of Earth.

$$T := (27.333)\cdot 24\cdot 60\cdot 60\cdot \text{sec} \qquad R := 6.368\cdot 10^6\cdot \text{m}$$

Using the given values of T, g, and R, we obtain the value of r, the distance between Earth and the Moon.

$$T = 2.362\cdot 10^6\ \cdot\text{sec} \qquad g := 9.8\cdot\frac{\text{m}}{\text{sec}^2}$$

$$r := \frac{1}{4}\cdot 4^{\left(\frac{2}{3}\right)}\cdot T^{\left(\frac{2}{3}\right)}\cdot g^{\left(\frac{1}{3}\right)}\cdot\frac{R^{\left(\frac{2}{3}\right)}}{\pi^{\left(\frac{2}{3}\right)}}$$

$$r = 3.829\cdot 10^8\ \cdot\text{m} \qquad r = 3.829\cdot 10^5\ \cdot\text{km}$$

EXERCISE 1.4 Repeat the example for another planet such as Mars or Venus.

Example 1.5

A small mass swings in a horizontal circle of radius r at the end of a string of length L_i and makes an angle θ_i with the vertical as shown in Figure 1.9. The string is slowly shortened by pulling it through a hole in its support. Write expressions for r, v, and time period of revolution T in terms of θ. Then graph these quantities.

Solution

We will use Eqs. (1.57) and (1.53), and $T = 2\pi/\omega$. Let us consider n = 25 values of θ from a very small angle to 90 degree angle. Radius r_1 (for n = 1) is 0.063 meter, while the vertical angle is 3.6 degrees. Three graphs are shown below.

(a) Which quantities become very small near θ = 0 degree angle and near θ = 90 degree angle, and why?

(b) Which quantities become very large at near θ = 0 degree angle and near θ = 90 degree angle, and why?

(c) What is the significance of the points where two graphs intersect?

$n := 1..25 \qquad \theta_n := \frac{\pi \cdot n}{50} \qquad g := 9.8$

$Li := 1.0 \qquad r0 := Li \cdot \sin(0) \qquad r0 = 0$

$r_n := Li \cdot \sin(\theta_n) \qquad v_n := \sqrt{\tan(\theta_n) \cdot g \cdot r_n}$

$T_n := \frac{2 \cdot \pi}{\left(\frac{v_n}{r_n}\right)} \qquad \theta_1 = 0.063 \cdot \text{rad} \qquad \theta_1 = 3.6 \cdot \text{deg}$

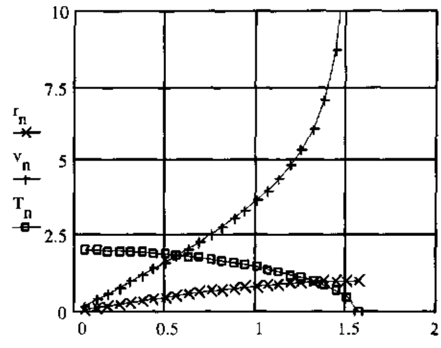
$\theta_{25} = 1.571 \cdot \text{rad} \qquad \theta_{25} = 90 \cdot \text{deg}$

$r_1 = 0.063 \qquad v_1 = 0.197 \qquad T_1 = 2.005$

$r_{10} = 0.588 \qquad v_{10} = 2.046 \qquad T_{10} = 1.805$

$r_{20} = 0.951 \qquad v_{20} = 5.356 \qquad T_{20} = 1.116$

$r_{25} = 1 \qquad v_{25} = 4.001 \cdot 10^{8} \qquad T_{25} = 1.571 \cdot 10^{-8}$



EXERCISE 1.5 Derive an expression for the tension Ft in the string as a function of the angle of the string with the vertical. Graph the tension Ft and angular frequency ω versus θ. Then answer all the questions in the example using this information.

PROBLEMS

1.1. The speed v of sound waves in air depends on the atmospheric pressure P and density ρ of the air. By using the method of dimensional analysis, find an expression for v in terms of P and ρ.

1.2. The velocity v of waves on a vibrating string depends on the tension T in the string and the mass per unit length λ of the string. Derive an expression for v by using the dimensional analysis method.

1.3. The time period T of a planet around the Sun of mass M is given by the following expression: $T^2 = 4\pi^2 r^3/MG$ where a is the radius of the circular orbit of the planet.

(a) What are the SI units of G?

(b) Derive the preceding expression by using the dimensional analysis method, that is, by assuming $T = T(r, M, G)$.

1.4. When a fluid flows in a pipe, the friction between the fluid and the surface of the pipe is given by the coefficient of viscosity η, defined by the equation $F/A = \eta(dv/ds)$, where F is the force of friction acting across an area A and dv/ds is the velocity gradient between layers of fluids.

(a) What are the units of η?

(b) If ΔP is the pressure difference and is directly proportional to Δl, by using the method of dimensional analysis, show that

$$\frac{\Delta P}{\Delta l} = \text{constant}\,\frac{\eta\phi}{r^4}$$

where ϕ is the volume flux of the fluid through the pipe and r is the radius of the pipe, as shown in Fig. P1.4.

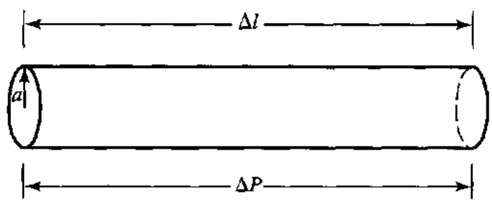


Figure P1.4

1.5. In Fig. 1.1, the mass of object C is 100 kg, the acceleration measured by the observer in the inertial system A is 100 m/s^2, while the observer in system B measures the acceleration as 90 m/s^2. What is the fictitious force? What is the acceleration of the noninertial system B? What could B do to achieve a true inertial system?

1.6. A mass m is given an initial velocity v_0 up an inclined plane of angle θ (θ is greater than the angle of friction). Find the distance the mass moves up the incline, the time it takes to reach this point, and the time it takes to return to its original position.

1.7. A box of mass m is connected by a rope that passes over a pulley to a box of mass M, as shown in Fig. P1.7. The coefficient of friction between m and the horizontal surface AB or the inclined surface BC is μ. Find the acceleration of the system and the tension in the rope for the portion whe the mass is moving (a) between A and B, and (b) between B and C.

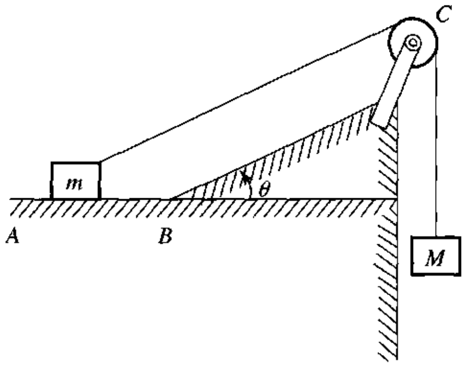


Figure P1.7

1.8. A man of mass M pushes horizontally a sled with a boy (sitting on it) of mass m. The coeffici of friction between the sled and the snow is μ, and the coefficient of friction between the man's feet and the snow is μ_s.

(a) Draw a clear diagram showing all the forces acting on the sled and the man.

(b) Calculate the horizontal and vertical components of the force when the man and the sled ha an acceleration a.

(c) What is the maximum acceleration the man can give to himself and the sled?

1.9. A man pushes a box of mass M with a force F using a stick AB of mass m and making an angle with the vertical, as shown in Fig. P1.9. The coefficient of friction between the box and the flo is μ.

(a) Draw a clear diagram showing all the forces.

(b) Calculate the value of F required to move the box with uniform velocity.

(c) Show that, if θ is less than the angle of friction, the box cannot be started by just pushing.

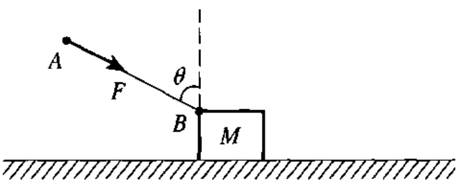


Figure P1.9

1.10. Consider a system of two masses and a pulley as shown in Fig. P1.10. Let $m_1 = 12$ kg, $m_2 = 8$ k the mass of the pulley $m = 10$ kg, and its radius $r = 10$ cm.

(a) Show all the forces acting on the system.

(b) Calculate T_1, T_2, and acceleration a. Assume the pulley to be a solid disk ($I_{disk} = mr^2/2$).

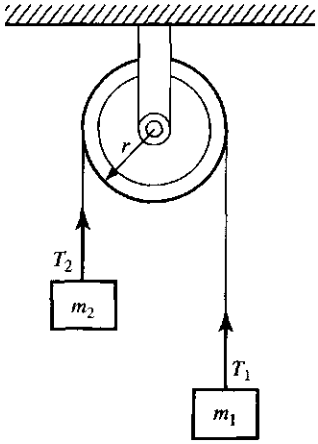


Figure P1.10

1.11. Repeat Problem 1.10 if the pulley were a hollow rim of the same mass and radius ($I_{\text{rim}} = mr^2$).

1.12. An automobile on a highway enters a curve of radius R and banking angle θ, as shown in Fig. P1.12. The coefficient of friction between the wheels and the road is μ. What are the maximum and minimum speeds with which a car can round the curve without skidding sideways?

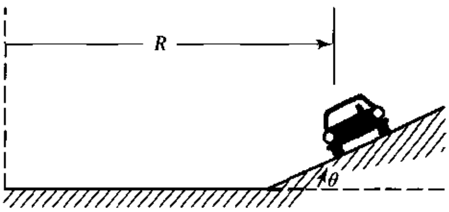


Figure P1.12

1.13. A mass M tied to a string of radius R is whirled in a vertical circle as shown in Fig. P1.13.
(**a**) Find the tension in the string at different points such as A, B, C, and D.
(**b**) What is the minimum velocity v_0 at the top point B so that the string won't slack?
(**c**) Graph T and v as a function of θ for given M and R.

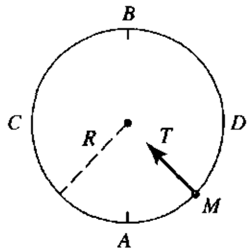


Figure P1.13

1.14. Two particles of masses m_1 and m_2 at a distance R from each other are under the influence of an attractive force F. If the two masses undergo uniform circular motion about each other with an angular velocity ω, show that

$$F = \left(\frac{m_1 m_2}{m_1 + m_2}\right)\omega^2 R = \frac{\omega^2 R}{(1/m_1) + (1/m_2)}$$

1.15. Calculate the height and velocity of a satellite that remains over the same point at all times as seen from Earth. Assume a circular orbit and express the height in terms of the radius of the Earth, R_e. Such a satellite, called a synchronous satellite, goes around Earth once every 24 h, so its position appears stationary with respect to a ground station. One such communication satellite was named Earlybird.

1.16. Consider a cone with an apex half-angle θ, as shown in Fig. P1.16. A particle of mass m slides without friction on the inside of the cone in a circular path in a horizontal plane with speed v. Draw a force diagram and calculate the radius of the circular path in terms of θ, v, and g.

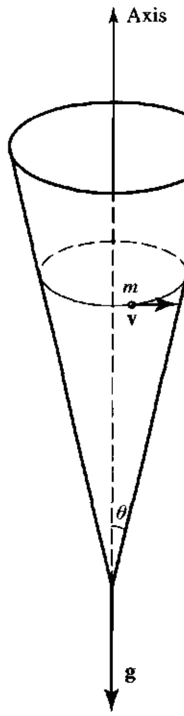


Figure P1.16

1.17. Repeat Problem 1.16 for the case when the surface is not frictionless and the coefficient of friction is μ.

1.18. Find the mass of the Sun, assuming that Earth moves in a circular orbit of radius 1.496×10^8 m and completes one revolution around the Sun in one year.

1.19. Find the distance between Earth and Mars by first calculating the distances of these planets from the Sun. The revolution period of Earth is 1.00 year and that of Mars is 1.88 years.

1.20. Repeat Problem 1.15 for a synchronous satellite going around Jupiter every 9 h, 50 min. Revolution period of Jupiter is 11.86 years, its mass is 317.80 times Earth's mass, and it's at a distance of 677.71×10^7 km from the Sun.

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