CHAPTER 7

FIRST-ORDER CIRCUITS

I often say that when you can measure what you are speaking about, and express it in numbers, you know something about it; but when you cannot express it in numbers, your knowledge is of a meager and unsatisfactory kind; it may be the beginning of knowledge, but you have scarcely, in your thoughts, advanced to the stage of a science, whatever the matter may be.

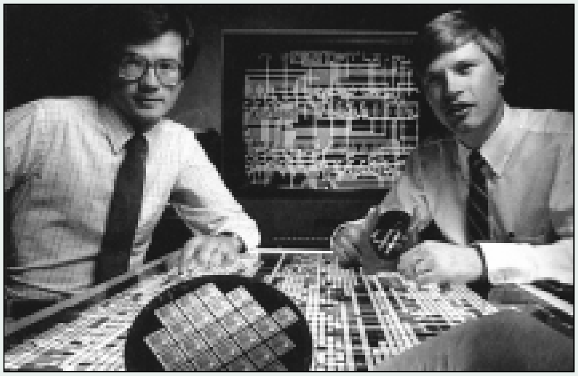
—Lord Kelvin

Enhancing Your Career

Careers in Computer Engineering Electrical engineering education has gone through drastic changes in recent decades. Most departments have come to be known as Department of Electrical and Computer Engineering, emphasizing the rapid changes due to computers. Computers occupy a prominent place in modern society and education. They have become commonplace and are helping to change the face of research, development, production, business, and entertainment. The scientist, engineer, doctor, attorney, teacher, airline pilot, businessperson—almost anyone benefits from a computer's abilities to store large amounts of information and to process that information in very short periods of time. The internet, a computer communication network, is becoming essential in business, education, and library science. Computer usage is growing by leaps and bounds.

Three major disciplines study computer systems: computer science, computer engineering, and information management science. Computer engineering has grown so fast and wide that it is divorcing itself from electrical engineering. But, in many schools of engineering, computer engineering is still an integral part of electrical engineering.

An education in computer engineering should provide breadth in software, hardware design, and basic modeling techniques. It should include courses in data structures, digital systems, computer architecture, microprocessors, interfacing, software engineering, and operating systems. Electrical engineers who specialize in computer engineering find jobs in computer industries and in numerous fields where computers are being used. Companies that produce software are growing rapidly in number and size and providing employment for those who are skilled in programming. An excellent way to advance one's knowledge of computers is to join the IEEE Computer Society, which sponsors diverse magazines, journals, and conferences.



Computer design of very large scale integrated (VLSI) circuits. Source: M. E. Hazen, Fundamentals of DC and AC Circuits, Philadelphia: Saunders, 1990, p. 20A4.

7.1 INTRODUCTION

Now that we have considered the three passive elements (resistors, capacitors, and inductors) and one active element (the op amp) individually, we are prepared to consider circuits that contain various combinations of two or three of the passive elements. In this chapter, we shall examine two types of simple circuits: a circuit comprising a resistor and capacitor and a circuit comprising a resistor and an inductor. These are called RC and RL circuits, respectively. As simple as these circuits are, they find continual applications in electronics, communications, and control systems, as we shall see.

We carry out the analysis of RC and RL circuits by applying Kirchhoff's laws, as we did for resistive circuits. The only difference is that applying Kirchhoff's laws to purely resistive circuits results in algebraic equations, while applying the laws to RC and RL circuits produces differential equations, which are more difficult to solve than algebraic equations. The differential equations resulting from analyzing RC and RL circuits are of the first order. Hence, the circuits are collectively known as *first-order* circuits.

> A first-order circuit is characterized by a first-order differential equation.

In addition to there being two types of first-order circuits (RC and RL), there are two ways to excite the circuits. The first way is by initial conditions of the storage elements in the circuits. In these so-called *source-free circuits*, we assume that energy is initially stored in the capacitive or inductive element. The energy causes current to flow in the circuit and is gradually dissipated in the resistors. Although source-free circuits are by definition free of independent sources, they may have dependent sources. The second way of exciting first-order circuits is by independent sources. In this chapter, the independent sources we will consider are dc sources. (In later chapters, we shall consider sinusoidal and exponential sources.) The two types of first-order circuits and the two ways of exciting them add up to the four possible situations we will study in this chapter.

Finally, we consider four typical applications of RC and RL circuits: delay and relay circuits, a photoflash unit, and an automobile ignition circuit.

7.2 THE SOURCE-FREE *RC* CIRCUIT

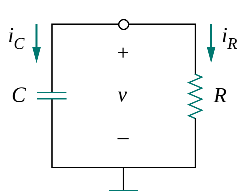


Figure 7.1 A source-free RC circuit.

A circuit response is the manner in which the circuit reacts to an excitation.

A source-free RC circuit occurs when its dc source is suddenly disconnected. The energy already stored in the capacitor is released to the resistors.

Consider a series combination of a resistor and an initially charged capacitor, as shown in Fig. 7.1. (The resistor and capacitor may be the equivalent resistance and equivalent capacitance of combinations of resistors and capacitors.) Our objective is to determine the circuit response, which, for pedagogic reasons, we assume to be the voltage $v(t)$ across the capacitor. Since the capacitor is initially charged, we can assume that at time $t = 0$, the initial voltage is

$$v(0) = V_0 \tag{7.1}$$

with the corresponding value of the energy stored as

$$w(0) = \frac{1}{2} C V_0^2 \tag{7.2}$$

Applying KCL at the top node of the circuit in Fig. 7.1,

$$i_C + i_R = 0 \tag{7.3}$$

By definition, $i_C = C\,dv/dt$ and $i_R = v/R$. Thus,

$$C\frac{dv}{dt} + \frac{v}{R} = 0 \tag{7.4a}$$

or

$$\frac{dv}{dt} + \frac{v}{RC} = 0 \tag{7.4b}$$

This is a *first-order differential equation*, since only the first derivative of v is involved. To solve it, we rearrange the terms as

$$\frac{dv}{v} = -\frac{1}{RC}\,dt \tag{7.5}$$

Integrating both sides, we get

$$\ln v = -\frac{t}{RC} + \ln A$$

where $\ln A$ is the integration constant. Thus,

$$\ln\frac{v}{A} = -\frac{t}{RC} \tag{7.6}$$

Taking powers of e produces

$$v(t) = Ae^{-t/RC}$$

But from the initial conditions, $v(0) = A = V_0$. Hence,

$$v(t) = V_0 e^{-t/RC} \tag{7.7}$$

This shows that the voltage response of the RC circuit is an exponential decay of the initial voltage. Since the response is due to the initial energy stored and the physical characteristics of the circuit and not due to some external voltage or current source, it is called the *natural response* of the circuit.

> The natural response of a circuit refers to the behavior (in terms of voltages and currents) of the circuit itself, with no external sources of excitation.

The natural response is illustrated graphically in Fig. 7.2. Note that at $t = 0$, we have the correct initial condition as in Eq. (7.1). As t increases, the voltage decreases toward zero. The rapidity with which the voltage decreases is expressed in terms of the *time constant*, denoted by the lower case Greek letter tau, τ.

The natural response depends on the nature of the circuit alone, with no external sources. In fact, the circuit has a response only because of the energy initially stored in the capacitor.

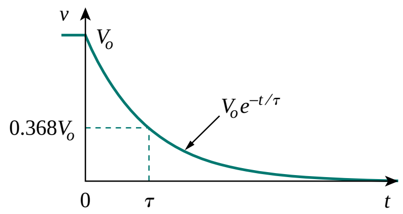


Figure 7.2 The voltage response of the RC circuit.

The **time constant** of a circuit is the time required for the response to decay by a factor of $1/e$ or 36.8 percent of its initial value.[1]

This implies that at $t = \tau$, Eq. (7.7) becomes

$$V_0 e^{-\tau/RC} = V_0 e^{-1} = 0.368 V_0$$

or

$$\boxed{\tau = RC} \tag{7.8}$$

In terms of the time constant, Eq. (7.7) can be written as

$$\boxed{v(t) = V_0 e^{-t/\tau}} \tag{7.9}$$

With a calculator it is easy to show that the value of $v(t)/V_0$ is as shown in Table 7.1. It is evident from Table 7.1 that the voltage $v(t)$ is less than 1 percent of V_0 after 5τ (five time constants). Thus, it is customary to assume that the capacitor is fully discharged (or charged) after five time constants. In other words, it takes 5τ for the circuit to reach its final state or steady state when no changes take place with time. Notice that for every time interval of τ, the voltage is reduced by 36.8 percent of its previous value, $v(t+\tau) = v(t)/e = 0.368v(t)$, regardless of the value of t.

TABLE 7.1 Values of $v(t)/V_0 = e^{-t/\tau}$.

t	$v(t)/V_0$
τ	0.36788
2τ	0.13534
3τ	0.04979
4τ	0.01832
5τ	0.00674

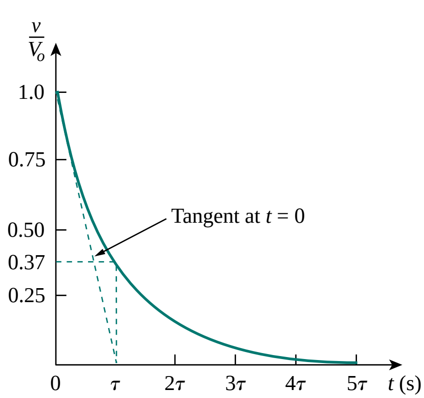


Figure 7.3 Graphical determination of the time constant τ from the response curve.

Observe from Eq. (7.8) that the smaller the time constant, the more rapidly the voltage decreases, that is, the faster the response. This is illustrated in Fig. 7.4. A circuit with a small time constant gives a fast response in that it reaches the steady state (or final state) quickly due to quick dissipation of energy stored, whereas a circuit with a large time

[1]The time constant may be viewed from another perspective. Evaluating the derivative of $v(t)$ in Eq. (7.7) at $t = 0$, we obtain

$$\frac{d}{dt}\left(\frac{v}{V_0}\right)\bigg|_{t=0} = -\frac{1}{\tau}e^{-t/\tau}\bigg|_{t=0} = -\frac{1}{\tau}$$

Thus the time constant is the initial rate of decay, or the time taken for v/V_0 to decay from unity to zero, assuming a constant rate of decay. This initial slope interpretation of the time constant is often used in the laboratory to find τ graphically from the response curve displayed on an oscilloscope. To find τ from the response curve, draw the tangent to the curve, as shown in Fig. 7.3. The tangent intercepts with the time axis at $t = \tau$.

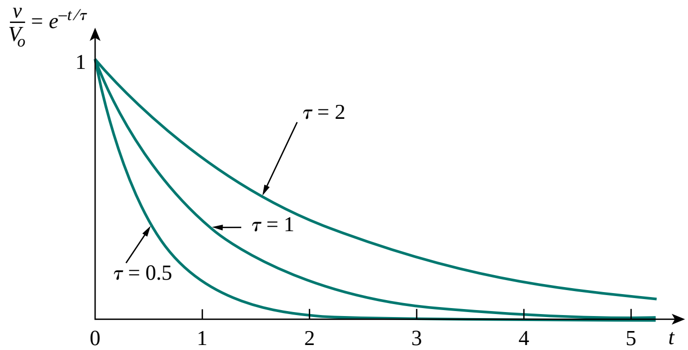


Figure 7.4 Plot of $v/V_0 = e^{-t/\tau}$ for various values of the time constant.

constant gives a slow response because it takes longer to reach steady state. At any rate, whether the time constant is small or large, the circuit reaches steady state in five time constants.

With the voltage $v(t)$ in Eq. (7.9), we can find the current $i_R(t)$,

$$i_R(t) = \frac{v(t)}{R} = \frac{V_0}{R} e^{-t/\tau} \tag{7.10}$$

The power dissipated in the resistor is

$$p(t) = v i_R = \frac{V_0^2}{R} e^{-2t/\tau} \tag{7.11}$$

The energy absorbed by the resistor up to time t is

$$\begin{aligned} w_R(t) &= \int_0^t p\,dt = \int_0^t \frac{V_0^2}{R} e^{-2t/\tau} dt \\ &= -\frac{\tau V_0^2}{2R} e^{-2t/\tau} \bigg|_0^t = \frac{1}{2} C V_0^2 (1 - e^{-2t/\tau}), \qquad \tau = RC \end{aligned} \tag{7.12}$$

Notice that as $t \to \infty$, $w_R(\infty) \to \frac{1}{2} C V_0^2$, which is the same as $w_C(0)$, the energy initially stored in the capacitor. The energy that was initially stored in the capacitor is eventually dissipated in the resistor.

In summary:

> **The Key to Working with a Source-free *RC* Circuit is Finding:**
>
> 1. The initial voltage $v(0) = V_0$ across the capacitor.
> 2. The time constant τ.

With these two items, we obtain the response as the capacitor voltage $v_C(t) = v(t) = v(0)e^{-t/\tau}$. Once the capacitor voltage is first obtained, other variables (capacitor current i_C, resistor voltage v_R, and resistor current i_R) can be determined. In finding the time constant $\tau = RC$, R is often the Thevenin equivalent resistance at the terminals of the capacitor; that is, we take out the capacitor C and find $R = R_{\text{Th}}$ at its terminals.

The time constant is the same regardless of what the output is defined to be.

When a circuit contains a single capacitor and several resistors and dependent sources, the Thevenin equivalent can be found at the terminals of the capacitor to form a simple *RC* circuit. Also, one can use Thevenin's theorem when several capacitors can be combined to form a single equivalent capacitor.

EXAMPLE 7.1

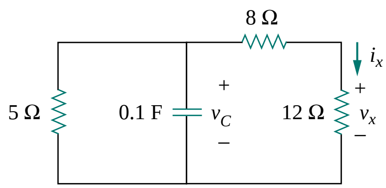


Figure 7.5 For Example 7.1.

In Fig. 7.5, let $v_C(0) = 15$ V. Find v_C, v_x, and i_x for $t > 0$.

Solution:

We first need to make the circuit in Fig. 7.5 conform with the standard RC circuit in Fig. 7.1. We find the equivalent resistance or the Thevenin resistance at the capacitor terminals. Our objective is always to first obtain capacitor voltage v_C. From this, we can determine v_x and i_x.

The 8-Ω and 12-Ω resistors in series can be combined to give a 20-Ω resistor. This 20-Ω resistor in parallel with the 5-Ω resistor can be combined so that the equivalent resistance is

$$R_{eq} = \frac{20 \times 5}{20 + 5} = 4\ \Omega$$

Hence, the equivalent circuit is as shown in Fig. 7.6, which is analogous to Fig. 7.1. The time constant is

$$\tau = R_{eq}C = 4(0.1) = 0.4 \text{ s}$$

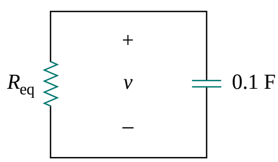


Figure 7.6 Equivalent circuit for the circuit in Fig. 7.5.

Thus,

$$v = v(0)e^{-t/\tau} = 15e^{-t/0.4} \text{ V}, \qquad v_C = v = 15e^{-2.5t} \text{ V}$$

From Fig. 7.5, we can use voltage division to get v_x; so

$$v_x = \frac{12}{12 + 8}v = 0.6(15e^{-2.5t}) = 9e^{-2.5t} \text{ V}$$

Finally,

$$i_x = \frac{v_x}{12} = 0.75e^{-2.5t} \text{ A}$$

PRACTICE PROBLEM 7.1

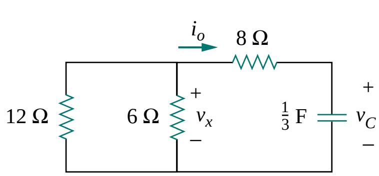


Figure 7.7 For Practice Prob. 7.1.

Refer to the circuit in Fig. 7.7. Let $v_C(0) = 30$ V. Determine v_C, v_x, and i_o for $t \geq 0$.

Answer: $30e^{-0.25t}$ V, $10e^{-0.25t}$ V, $-2.5e^{-0.25t}$ A.

EXAMPLE 7.2

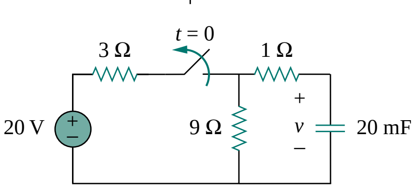


Figure 7.8 For Example 7.2.

The switch in the circuit in Fig. 7.8 has been closed for a long time, and it is opened at $t = 0$. Find $v(t)$ for $t \geq 0$. Calculate the initial energy stored in the capacitor.

Solution:

For $t < 0$, the switch is closed; the capacitor is an open circuit to dc, as represented in Fig. 7.9(a). Using voltage division

$$v_C(t) = \frac{9}{9+3}(20) = 15 \text{ V}, \qquad t < 0$$

Since the voltage across a capacitor cannot change instantaneously, the voltage across the capacitor at $t = 0^-$ is the same at $t = 0$, or

$$v_C(0) = V_0 = 15 \text{ V}$$

For $t > 0$, the switch is opened, and we have the RC circuit shown in Fig. 7.9(b). [Notice that the RC circuit in Fig. 7.9(b) is source free; the independent source in Fig. 7.8 is needed to provide V_0 or the initial energy in the capacitor.] The 1-Ω and 9-Ω resistors in series give

$$R_{\text{eq}} = 1 + 9 = 10\ \Omega$$

The time constant is

$$\tau = R_{\text{eq}}C = 10 \times 20 \times 10^{-3} = 0.2 \text{ s}$$

Thus, the voltage across the capacitor for $t \geq 0$ is

$$v(t) = v_C(0)e^{-t/\tau} = 15e^{-t/0.2} \text{ V}$$

or

$$v(t) = 15e^{-5t} \text{ V}$$

The initial energy stored in the capacitor is

$$w_C(0) = \frac{1}{2}Cv_C^2(0) = \frac{1}{2} \times 20 \times 10^{-3} \times 15^2 = 2.25 \text{ J}$$

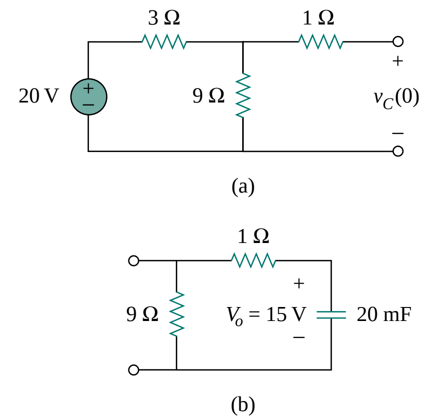


Figure 7.9 For Example 7.2: (a) $t < 0$, (b) $t > 0$.

PRACTICE PROBLEM 7.2

If the switch in Fig. 7.10 opens at $t = 0$, find $v(t)$ for $t \geq 0$ and $w_C(0)$.

Answer: $8e^{-2t}$ V, 5.33 J.

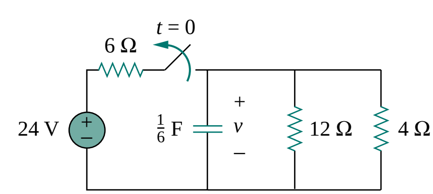


Figure 7.10 For Practice Prob. 7.2.

7.3 THE SOURCE-FREE *RL* CIRCUIT

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Consider the series connection of a resistor and an inductor, as shown in Fig. 7.11. Our goal is to determine the circuit response, which we will assume to be the current $i(t)$ through the inductor. We select the inductor current as the response in order to take advantage of the idea that the inductor current cannot change instantaneously. At $t = 0$, we assume that the inductor has an initial current I_0, or

$$i(0) = I_0 \tag{7.13}$$

with the corresponding energy stored in the inductor as

$$w(0) = \frac{1}{2}LI_0^2 \tag{7.14}$$

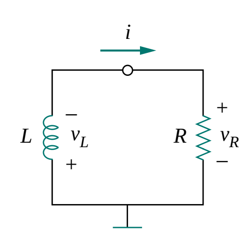


Figure 7.11 A source-free RL circuit.

Applying KVL around the loop in Fig. 7.11,

$$v_L + v_R = 0 \tag{7.15}$$

But $v_L = L\,di/dt$ and $v_R = iR$. Thus,

$$L\frac{di}{dt} + Ri = 0$$

or

$$\frac{di}{dt} + \frac{R}{L}i = 0 \tag{7.16}$$

Rearranging terms and integrating gives

$$\int_{I_0}^{i(t)} \frac{di}{i} = -\int_0^t \frac{R}{L}dt$$

$$\ln i\Big|_{I_0}^{i(t)} = -\frac{Rt}{L}\Big|_0^t \quad \Longrightarrow \quad \ln i(t) - \ln I_0 = -\frac{Rt}{L} + 0$$

or

$$\ln\frac{i(t)}{I_0} = -\frac{Rt}{L} \tag{7.17}$$

Taking the powers of e, we have

$$i(t) = I_0 e^{-Rt/L} \tag{7.18}$$

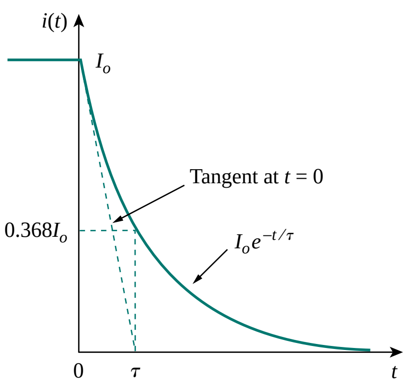


Figure 7.12 The current response of the RL circuit.

This shows that the natural response of the RL circuit is an exponential decay of the initial current. The current response is shown in Fig. 7.12. It is evident from Eq. (7.18) that the time constant for the RL circuit is

$$\boxed{\tau = \frac{L}{R}} \tag{7.19}$$

with τ again having the unit of seconds. Thus, Eq. (7.18) may be written as

$$\boxed{i(t) = I_0 e^{-t/\tau}} \tag{7.20}$$

The smaller the time constant τ of a circuit, the faster the rate of decay of the response. The larger the time constant, the slower the rate of decay of the response. At any rate, the response decays to less than 1 percent of its initial value (i.e., reaches steady state) after 5τ.

Figure 7.12 shows an initial slope interpretation may be given to τ.

With the current in Eq. (7.20), we can find the voltage across the resistor as

$$v_R(t) = iR = I_0 R e^{-t/\tau} \tag{7.21}$$

The power dissipated in the resistor is

$$p = v_R i = I_0^2 R e^{-2t/\tau} \tag{7.22}$$

The energy absorbed by the resistor is

$$w_R(t) = \int_0^t p\,dt = \int_0^t I_0^2 R e^{-2t/\tau}\,dt = -\frac{1}{2}\tau I_0^2 R e^{-2t/\tau}\Big|_0^t, \qquad \tau = \frac{L}{R}$$

or

$$w_R(t) = \frac{1}{2}LI_0^2(1 - e^{-2t/\tau}) \tag{7.23}$$

Note that as $t \to \infty$, $w_R(\infty) \to \frac{1}{2}LI_0^2$, which is the same as $w_L(0)$, the initial energy stored in the inductor as in Eq. (7.14). Again, the energy initially stored in the inductor is eventually dissipated in the resistor.

In summary:

The Key to Working with a Source-free *RL* Circuit is to Find:

1. The initial current $i(0) = I_0$ through the inductor.
2. The time constant τ of the circuit.

With the two items, we obtain the response as the inductor current $i_L(t) = i(t) = i(0)e^{-t/\tau}$. Once we determine the inductor current i_L, other variables (inductor voltage v_L, resistor voltage v_R, and resistor current i_R) can be obtained. Note that in general, R in Eq. (7.19) is the Thevenin resistance at the terminals of the inductor.

When a circuit has a single inductor and several resistors and dependent sources, the Thevenin equivalent can be found at the terminals of the inductor to form a simple *RL* circuit. Also, one can use Thevenin's theorem when several inductors can be combined to form a single equivalent inductor.

EXAMPLE 7.3

Assuming that $i(0) = 10$ A, calculate $i(t)$ and $i_x(t)$ in the circuit in Fig. 7.13.

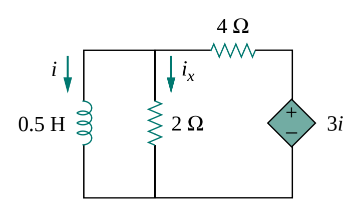


Figure 7.13 For Example 7.3.

Solution:

There are two ways we can solve this problem. One way is to obtain the equivalent resistance at the inductor terminals and then use Eq. (7.20). The other way is to start from scratch by using Kirchhoff's voltage law. Whichever approach is taken, it is always better to first obtain the inductor current.

METHOD 1 The equivalent resistance is the same as the Thevenin resistance at the inductor terminals. Because of the dependent source, we insert a voltage source with $v_o = 1$ V at the inductor terminals a-b, as in Fig. 7.14(a). (We could also insert a 1-A current source at the terminals.) Applying KVL to the two loops results in

$$2(i_1 - i_2) + 1 = 0 \quad \Longrightarrow \quad i_1 - i_2 = -\frac{1}{2} \tag{7.3.1}$$

$$6i_2 - 2i_1 - 3i_1 = 0 \quad \Longrightarrow \quad i_2 = \frac{5}{6}i_1 \tag{7.3.2}$$

Substituting Eq. (7.3.2) into Eq. (7.3.1) gives

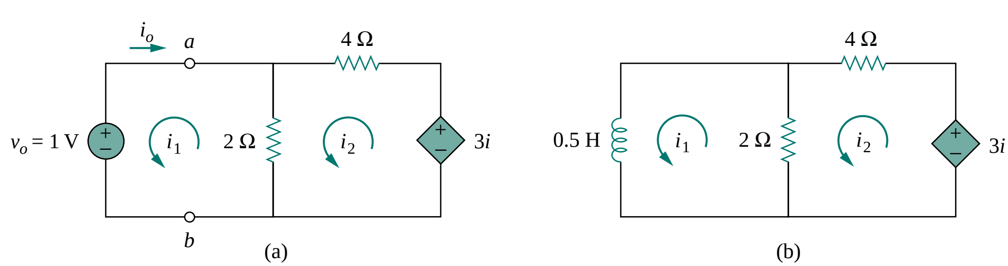


Figure 7.14 Solving the circuit in Fig. 7.13.

$$i_1 = -3 \text{ A}, \qquad i_o = -i_1 = 3 \text{ A}$$

Hence,

$$R_{\text{eq}} = R_{\text{Th}} = \frac{v_o}{i_o} = \frac{1}{3}\ \Omega$$

The time constant is

$$\tau = \frac{L}{R_{\text{eq}}} = \frac{\frac{1}{2}}{\frac{1}{3}} = \frac{3}{2}\text{ s}$$

Thus, the current through the inductor is

$$i(t) = i(0)e^{-t/\tau} = 10e^{-(2/3)t} \text{ A}, \qquad t > 0$$

METHOD 2 We may directly apply KVL to the circuit as in Fig. 7.14(b). For loop 1,

$$\frac{1}{2}\frac{di_1}{dt} + 2(i_1 - i_2) = 0$$

or

$$\frac{di_1}{dt} + 4i_1 - 4i_2 = 0 \tag{7.3.3}$$

For loop 2,

$$6i_2 - 2i_1 - 3i_1 = 0 \qquad \Longrightarrow \qquad i_2 = \frac{5}{6}i_1 \tag{7.3.4}$$

Substituting Eq. (7.3.4) into Eq. (7.3.3) gives

$$\frac{di_1}{dt} + \frac{2}{3}i_1 = 0$$

Rearranging terms,

$$\frac{di_1}{i_1} = -\frac{2}{3}dt$$

Since $i_1 = i$, we may replace i_1 with i and integrate:

$$\ln i \Big|_{i(0)}^{i(t)} = -\frac{2}{3}t \Big|_0^t$$

or

$$\ln \frac{i(t)}{i(0)} = -\frac{2}{3}t$$

Taking the powers of e, we finally obtain

$$i(t) = i(0)e^{-(2/3)t} = 10e^{-(2/3)t} \text{ A}, \qquad t > 0$$

which is the same as by Method 1.

The voltage across the inductor is

$$v = L\frac{di}{dt} = 0.5(10)\left(-\frac{2}{3}\right)e^{-(2/3)t} = -\frac{10}{3}e^{-(2/3)t} \text{ V}$$

Since the inductor and the 2-Ω resistor are in parallel,

$$i_x(t) = \frac{v}{2} = -1.667e^{-(2/3)t} \text{ A}, \qquad t > 0$$

PRACTICE PROBLEM 7.3

Find i and v_x in the circuit in Fig. 7.15. Let $i(0) = 5$ A.

Answer: $5e^{-53t}$ A, $-15e^{-53t}$ V.

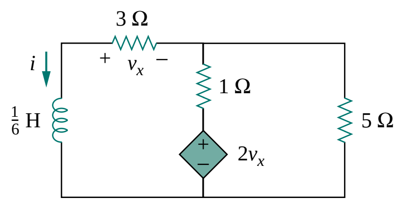


Figure 7.15 For Practice Prob. 7.3.

EXAMPLE 7.4

The switch in the circuit of Fig. 7.16 has been closed for a long time. At $t = 0$, the switch is opened. Calculate $i(t)$ for $t > 0$.

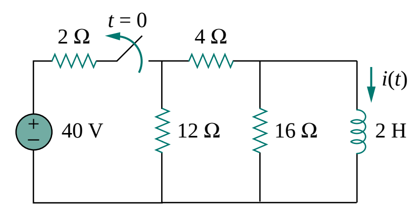


Figure 7.16 For Example 7.4.

Solution:

When $t < 0$, the switch is closed, and the inductor acts as a short circuit to dc. The 16-Ω resistor is short-circuited; the resulting circuit is shown in Fig. 7.17(a). To get i_1 in Fig. 7.17(a), we combine the 4-Ω and 12-Ω resistors in parallel to get

$$\frac{4 \times 12}{4 + 12} = 3\ \Omega$$

Hence,

$$i_1 = \frac{40}{2 + 3} = 8\ \text{A}$$

We obtain $i(t)$ from i_1 in Fig. 7.17(a) using current division, by writing

$$i(t) = \frac{12}{12 + 4} i_1 = 6\ \text{A}, \qquad t < 0$$

Since the current through an inductor cannot change instantaneously,

$$i(0) = i(0^-) = 6\ \text{A}$$

When $t > 0$, the switch is open and the voltage source is disconnected. We now have the RL circuit in Fig. 7.17(b). Combining the resistors, we have

$$R_{\text{eq}} = (12 + 4) \parallel 16 = 8\ \Omega$$

The time constant is

$$\tau = \frac{L}{R_{\text{eq}}} = \frac{2}{8} = \frac{1}{4}\ \text{s}$$

Thus,

$$i(t) = i(0)e^{-t/\tau} = 6e^{-4t}\ \text{A}$$

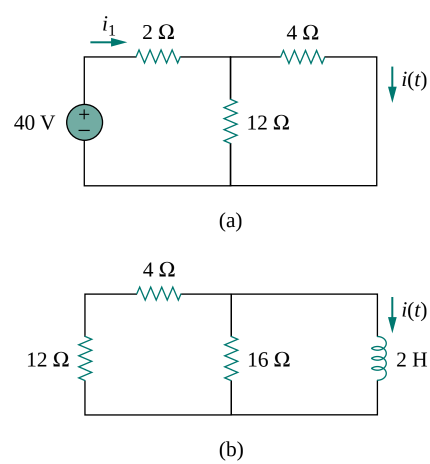


Figure 7.17 Solving the circuit of Fig. 7.16: (a) for $t < 0$, (b) for $t > 0$.

PRACTICE PROBLEM 7.4

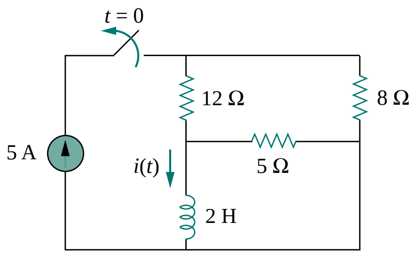


Figure 7.18 For Practice Prob. 7.4.

For the circuit in Fig. 7.18, find $i(t)$ for $t > 0$.

Answer: $2e^{-2t}$ A, $t > 0$.

EXAMPLE 7.5

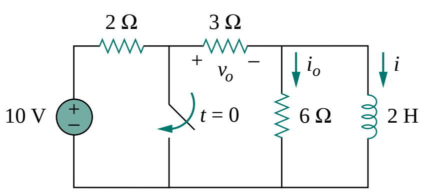


Figure 7.19 For Example 7.5.

In the circuit shown in Fig. 7.19, find i_o, v_o, and i for all time, assuming that the switch was open for a long time.

Solution:

It is better to first find the inductor current i and then obtain other quantities from it.

For $t < 0$, the switch is open. Since the inductor acts like a short circuit to dc, the 6-Ω resistor is short-circuited, so that we have the circuit shown in Fig. 7.20(a). Hence, $i_o = 0$, and

$$i(t) = \frac{10}{2+3} = 2 \text{ A}, \qquad t < 0$$

$$v_o(t) = 3i(t) = 6 \text{ V}, \qquad t < 0$$

Thus, $i(0) = 2$.

For $t > 0$, the switch is closed, so that the voltage source is short-circuited. We now have a source-free RL circuit as shown in Fig. 7.20(b). At the inductor terminals,

$$R_{\text{Th}} = 3 \parallel 6 = 2\ \Omega$$

so that the time constant is

$$\tau = \frac{L}{R_{\text{Th}}} = 1 \text{ s}$$

Hence,

$$i(t) = i(0)e^{-t/\tau} = 2e^{-t} \text{ A}, \qquad t > 0$$

Since the inductor is in parallel with the 6-Ω and 3-Ω resistors,

$$v_o(t) = -v_L = -L\frac{di}{dt} = -2(-2e^{-t}) = 4e^{-t} \text{ V}, \qquad t > 0$$

and

$$i_o(t) = \frac{v_L}{6} = -\frac{2}{3}e^{-t} \text{ A}, \qquad t > 0$$

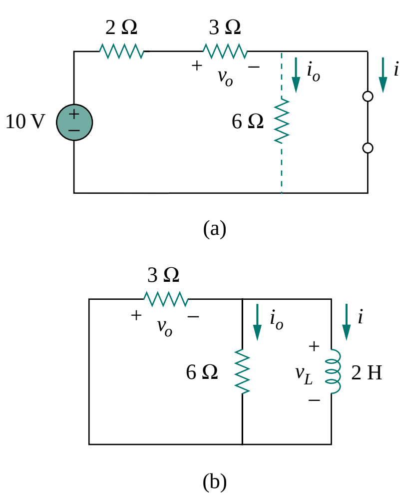


Figure 7.20 The circuit in Fig. 7.19 for: (a) $t < 0$, (b) $t > 0$.

Thus, for all time,

$$i_o(t) = \begin{cases} 0\text{ A}, & t < 0 \\ -\dfrac{2}{3}e^{-t}\text{ A}, & t > 0 \end{cases}, \qquad v_o(t) = \begin{cases} 6\text{ V}, & t < 0 \\ 4e^{-t}\text{ V}, & t > 0 \end{cases}$$

$$i(t) = \begin{cases} 2\text{ A}, & t < 0 \\ 2e^{-t}\text{ A}, & t \geq 0 \end{cases}$$

We notice that the inductor current is continuous at $t = 0$, while the current through the 6-Ω resistor drops from 0 to $-2/3$ at $t = 0$, and the voltage across the 3-Ω resistor drops from 6 to 4 at $t = 0$. We also notice that the time constant is the same regardless of what the output is defined to be. Figure 7.21 plots i and i_o.

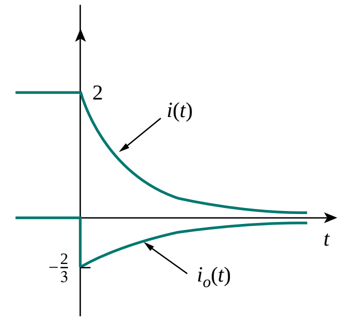


Figure 7.21 A plot of i and i_0.

PRACTICE PROBLEM 7.5

Determine i, i_o, and v_o for all t in the circuit shown in Fig. 7.22. Assume that the switch was closed for a long time.

Answer: $i = \begin{cases} 4\text{ A}, & t < 0 \\ 4e^{-2t}\text{ A}, & t \geq 0 \end{cases}$, $i_o = \begin{cases} 2\text{ A}, & t < 0 \\ -(4/3)e^{-2t}\text{ A}, & t > 0 \end{cases}$,

$$v_o = \begin{cases} 4\text{ V}, & t < 0 \\ -(8/3)e^{-2t}\text{ V}, & t > 0 \end{cases}$$

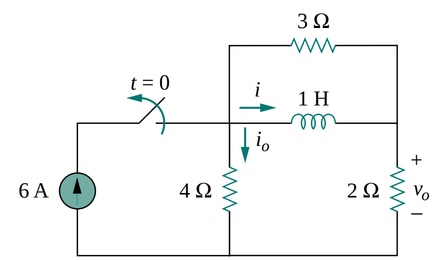


Figure 7.22 For Practice Prob. 7.5.

7.4 SINGULARITY FUNCTIONS

Before going on with the second half of this chapter, we need to digress and consider some mathematical concepts that will aid our understanding of transient analysis. A basic understanding of singularity functions will help us make sense of the response of first-order circuits to a sudden application of an independent dc voltage or current source.

Singularity functions (also called *switching functions*) are very useful in circuit analysis. They serve as good approximations to the switching signals that arise in circuits with switching operations. They are helpful in the neat, compact description of some circuit phenomena, especially the step response of RC or RL circuits to be discussed in the next sections. By definition,

> Singularity functions are functions that either are discontinuous or have discontinuous derivatives.

The three most widely used singularity functions in circuit analysis are the *unit step*, the *unit impulse*, and the *unit ramp* functions.

The **unit step function** $u(t)$ is 0 for negative values of t and 1 for positive values of t.

In mathematical terms,

$$\boxed{u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}} \tag{7.24}$$

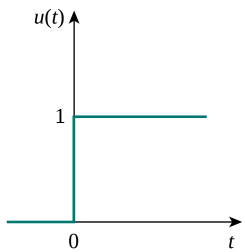


Figure 7.23 The unit step function.

The unit step function is undefined at $t = 0$, where it changes abruptly from 0 to 1. It is dimensionless, like other mathematical functions such as sine and cosine. Figure 7.23 depicts the unit step function. If the abrupt change occurs at $t = t_0$ (where $t_0 > 0$) instead of $t = 0$, the unit step function becomes

$$u(t - t_0) = \begin{cases} 0, & t < t_0 \\ 1, & t > t_0 \end{cases} \tag{7.25}$$

which is the same as saying that $u(t)$ is delayed by t_0 seconds, as shown in Fig. 7.24(a). To get Eq. (7.25) from Eq. (7.24), we simply replace every t by $t - t_0$. If the change is at $t = -t_0$, the unit step function becomes

$$u(t + t_0) = \begin{cases} 0, & t < -t_0 \\ 1, & t > -t_0 \end{cases} \tag{7.26}$$

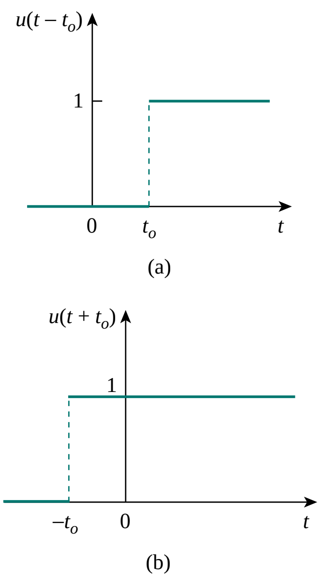


Figure 7.24 (a) The unit step function delayed by t_0, (b) the unit step advanced by t_0.

meaning that $u(t)$ is advanced by t_0 seconds, as shown in Fig. 7.24(b).

We use the step function to represent an abrupt change in voltage or current, like the changes that occur in the circuits of control systems and digital computers. For example, the voltage

$$v(t) = \begin{cases} 0, & t < t_0 \\ V_0, & t > t_0 \end{cases} \tag{7.27}$$

may be expressed in terms of the unit step function as

$$v(t) = V_0 u(t - t_0) \tag{7.28}$$

Alternatively, we may derive Eqs. (7.25) and (7.26) from Eq. (7.24) by writing $u[f(t)] = 1$, $f(t) > 0$, where $f(t)$ may be $t - t_0$ or $t + t_0$.

If we let $t_0 = 0$, then $v(t)$ is simply the step voltage $V_0 u(t)$. A voltage source of $V_0 u(t)$ is shown in Fig. 7.25(a); its equivalent circuit is shown in Fig. 7.25(b). It is evident in Fig. 7.25(b) that terminals a-b are short-circuited ($v = 0$) for $t < 0$ and that $v = V_0$ appears at the terminals for $t > 0$. Similarly, a current source of $I_0 u(t)$ is shown in Fig. 7.26(a), while its equivalent circuit is in Fig. 7.26(b). Notice that for $t < 0$, there is an open circuit ($i = 0$), and that $i = I_0$ flows for $t > 0$.

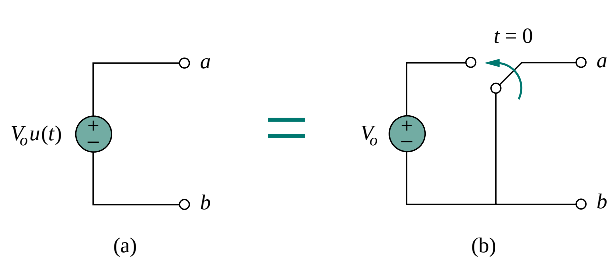


Figure 7.25 (a) Voltage source of $V_0 u(t)$, (b) its equivalent circuit.

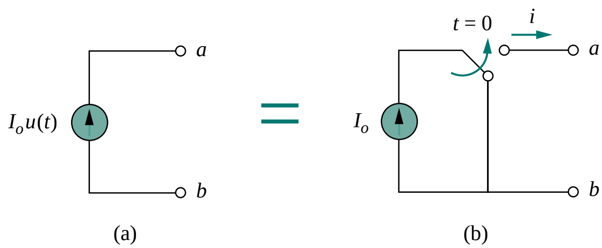


Figure 7.26 (a) Current source of $I_0 u(t)$, (b) its equivalent circuit.

The derivative of the unit step function $u(t)$ is the *unit impulse function* $\delta(t)$, which we write as

$$\boxed{\delta(t) = \frac{d}{dt}u(t) = \begin{cases} 0, & t < 0 \\ \text{Undefined}, & t = 0 \\ 0, & t > 0 \end{cases}} \tag{7.29}$$

The unit impulse function—also known as the *delta* function—is shown in Fig. 7.27.

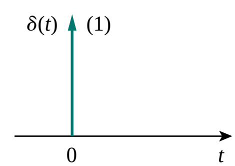


Figure 7.27 The unit impulse function.

> The unit impulse function $\delta(t)$ is zero everywhere except at $t = 0$, where it is undefined.

Impulsive currents and voltages occur in electric circuits as a result of switching operations or impulsive sources. Although the unit impulse function is not physically realizable (just like ideal sources, ideal resistors, etc.), it is a very useful mathematical tool.

The unit impulse may be regarded as an applied or resulting shock. It may be visualized as a very short duration pulse of unit area. This may be expressed mathematically as

$$\int_{0^-}^{0^+} \delta(t)\, dt = 1 \tag{7.30}$$

where $t = 0^-$ denotes the time just before $t = 0$ and $t = 0^+$ is the time just after $t = 0$. For this reason, it is customary to write 1 (denoting unit area) beside the arrow that is used to symbolize the unit impulse function, as in Fig. 7.27. The unit area is known as the *strength* of the impulse function. When an impulse function has a strength other than unity, the area of the impulse is equal to its strength. For example, an impulse function $10\delta(t)$ has an area of 10. Figure 7.28 shows the impulse functions $5\delta(t+2)$, $10\delta(t)$, and $-4\delta(t-3)$.

To illustrate how the impulse function affects other functions, let us evaluate the integral

$$\int_{a}^{b} f(t)\delta(t - t_0)\, dt \tag{7.31}$$

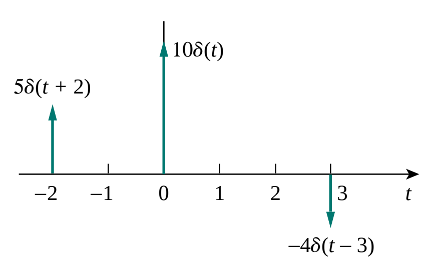


Figure 7.28 Three impulse functions.

where $a < t_0 < b$. Since $\delta(t - t_0) = 0$ except at $t = t_0$, the integrand is zero except at t_0. Thus,

$$\int_a^b f(t)\delta(t-t_0)\,dt = \int_a^b f(t_0)\delta(t-t_0)\,dt$$
$$= f(t_0)\int_a^b \delta(t-t_0)\,dt = f(t_0)$$

or

$$\boxed{\int_a^b f(t)\delta(t-t_0)\,dt = f(t_0)} \tag{7.32}$$

This shows that when a function is integrated with the impulse function, we obtain the value of the function at the point where the impulse occurs. This is a highly useful property of the impulse function known as the *sampling* or *sifting* property. The special case of Eq. (7.31) is for $t_0 = 0$. Then Eq. (7.32) becomes

$$\int_{0^-}^{0^+} f(t)\delta(t)\,dt = f(0) \tag{7.33}$$

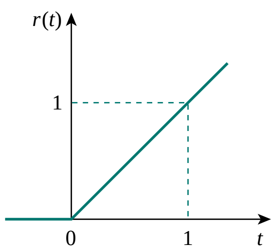


Figure 7.29 The unit ramp function.

Integrating the unit step function $u(t)$ results in the *unit ramp function* $r(t)$; we write

$$r(t) = \int_{-\infty}^{t} u(t)\,dt = tu(t) \tag{7.34}$$

or

$$\boxed{r(t) = \begin{cases} 0, & t \le 0 \\ t, & t \ge 0 \end{cases}} \tag{7.35}$$

> The unit ramp function is zero for negative values of t and has a unit slope for positive values of t.

Figure 7.29 shows the unit ramp function. In general, a ramp is a function that changes at a constant rate.

The unit ramp function may be delayed or advanced as shown in Fig. 7.30. For the delayed unit ramp function,

$$r(t-t_0) = \begin{cases} 0, & t \le t_0 \\ t-t_0, & t \ge t_0 \end{cases} \tag{7.36}$$

and for the advanced unit ramp function,

$$r(t+t_0) = \begin{cases} 0, & t \le -t_0 \\ t-t_0, & t \ge -t_0 \end{cases} \tag{7.37}$$

We should keep in mind that the three singularity functions (impulse, step, and ramp) are related by differentiation as

$$\delta(t) = \frac{du(t)}{dt}, \qquad u(t) = \frac{dr(t)}{dt} \tag{7.38}$$

or by integration as

$$u(t) = \int_{-\infty}^{t} \delta(t)\,dt, \qquad r(t) = \int_{-\infty}^{t} u(t)\,dt \tag{7.39}$$

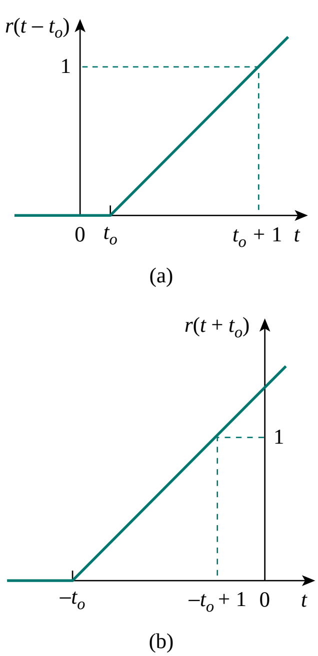


Figure 7.30 The unit ramp function: (a) delayed by t_0, (b) advanced by t_0.

Although there are many more singularity functions, we are only interested in these three (the impulse function, the unit step function, and the ramp function) at this point.

EXAMPLE 7.6

Express the voltage pulse in Fig. 7.31 in terms of the unit step. Calculate its derivative and sketch it.

Solution:

The type of pulse in Fig. 7.31 is called the *gate function*. It may be regarded as a step function that switches on at one value of t and switches off at another value of t. The gate function shown in Fig. 7.31 switches on at $t = 2$ s and switches off at $t = 5$ s. It consists of the sum of two unit step functions as shown in Fig. 7.32(a). From the figure, it is evident that

$$v(t) = 10u(t-2) - 10u(t-5) = 10[u(t-2) - u(t-5)]$$

Taking the derivative of this gives

$$\frac{dv}{dt} = 10[\delta(t-2) - \delta(t-5)]$$

which is shown in Fig. 7.32(b). We can obtain Fig. 7.32(b) directly from Fig. 7.31 by simply observing that there is a sudden increase by 10 V at $t = 2$ s leading to $10\delta(t-2)$. At $t = 5$ s, there is a sudden decrease by 10 V leading to -10 V $\delta(t-5)$.

Gate functions are used along with switches to pass or block another signal.

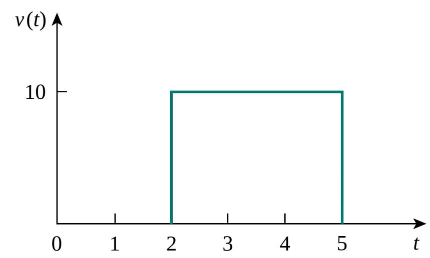


Figure 7.31 For Example 7.6.

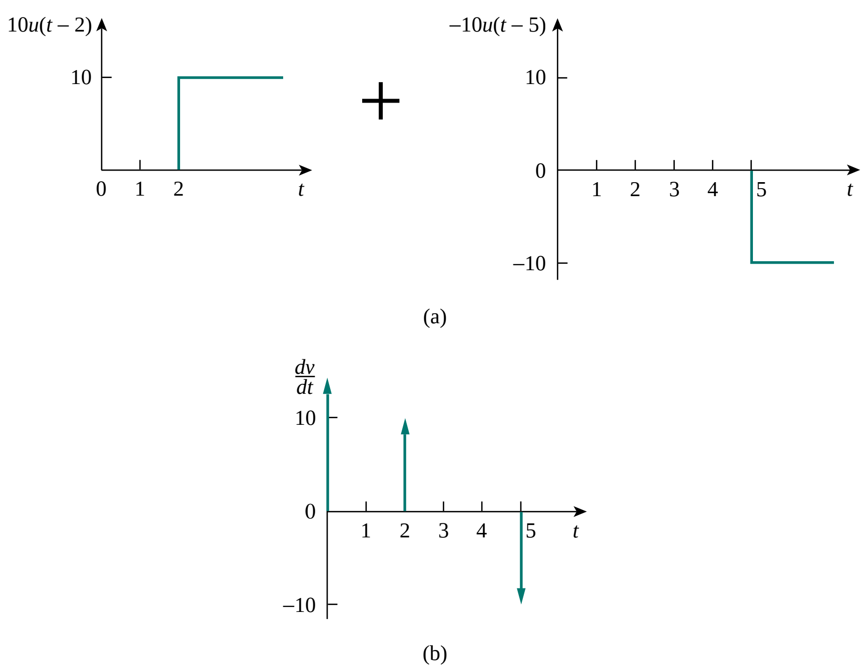


Figure 7.32 (a) Decomposition of the pulse in Fig. 7.31, (b) derivative of the pulse in Fig. 7.31.

PRACTICE PROBLEM 7.6

Express the current pulse in Fig. 7.33 in terms of the unit step. Find its integral and sketch it.

Answer: $10[u(t)-2u(t-2)+u(t-4)]$, $10[r(t)-2r(t-2)+r(t-4)]$. See Fig. 7.34.

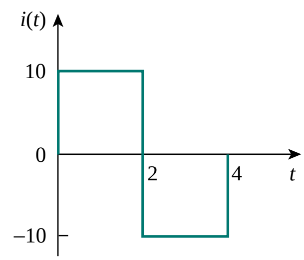


Figure 7.33 For Practice Prob. 7.6.

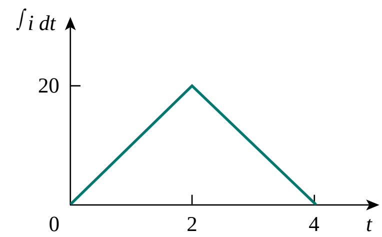


Figure 7.34 Integral of $i(t)$ in Fig. 7.33.

EXAMPLE 7.7

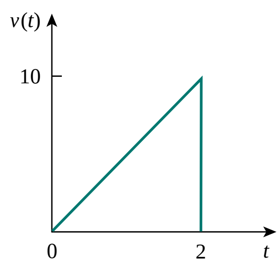


Figure 7.35 For Example 7.7.

Express the *sawtooth* function shown in Fig. 7.35 in terms of singularity functions.

Solution:

There are three ways of solving this problem. The first method is by mere observation of the given function, while the other methods involve some graphical manipulations of the function.

METHOD 1 By looking at the sketch of $v(t)$ in Fig. 7.35, it is not hard to notice that the given function $v(t)$ is a combination of singularity functions. So we let

$$v(t) = v_1(t) + v_2(t) + \cdots \tag{7.7.1}$$

The function $v_1(t)$ is the ramp function of slope 5, shown in Fig. 7.36(a); that is,

$$v_1(t) = 5r(t) \tag{7.7.2}$$

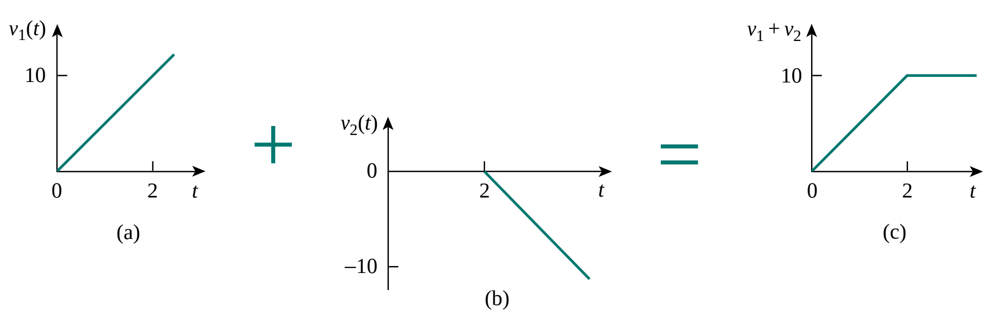


Figure 7.36 Partial decomposition of $v(t)$ in Fig. 7.35.

Since $v_1(t)$ goes to infinity, we need another function at $t = 2$ s in order to get $v(t)$. We let this function be v_2, which is a ramp function of slope -5, as shown in Fig. 7.36(b); that is,

$$v_2(t) = -5r(t-2) \tag{7.7.3}$$

Adding v_1 and v_2 gives us the signal in Fig. 7.36(c). Obviously, this is not the same as $v(t)$ in Fig. 7.35. But the difference is simply a constant 10 units for $t > 2$ s. By adding a third signal v_3, where

$$v_3 = -10u(t-2) \tag{7.7.4}$$

we get $v(t)$, as shown in Fig. 7.37. Substituting Eqs. (7.7.2) through (7.7.4) into Eq. (7.7.1) gives

$$v(t) = 5r(t) - 5r(t-2) - 10u(t-2)$$

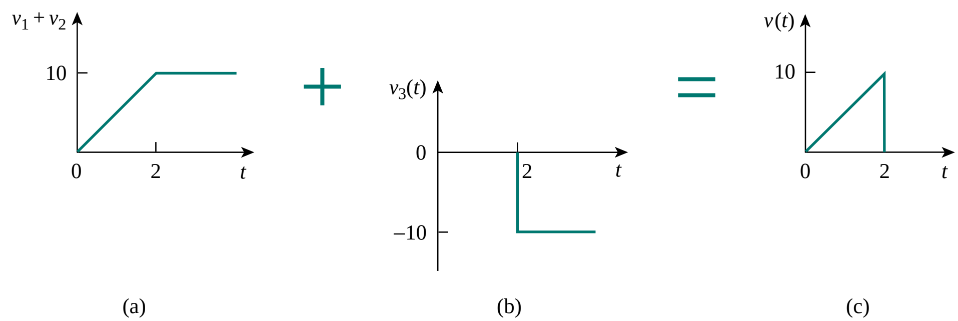


Figure 7.37 Complete decomposition of $v(t)$ in Fig. 7.35.

METHOD 2 A close observation of Fig. 7.35 reveals that $v(t)$ is a multiplication of two functions: a ramp function and a gate function. Thus,

$$\begin{aligned} v(t) &= 5t[u(t) - u(t-2)] \\ &= 5tu(t) - 5tu(t-2) \\ &= 5r(t) - 5(t-2+2)u(t-2) \\ &= 5r(t) - 5(t-2)u(t-2) - 10u(t-2) \\ &= 5r(t) - 5r(t-2) - 10u(t-2) \end{aligned}$$

the same as before.

METHOD 3 This method is similar to Method 2. We observe from Fig. 7.35 that $v(t)$ is a multiplication of a ramp function and a unit step function, as shown in Fig. 7.38. Thus,

$$v(t) = 5r(t)u(-t+2)$$

If we replace $u(-t)$ by $1 - u(t)$, then we can replace $u(-t+2)$ by $1 - u(t-2)$. Hence,

$$v(t) = 5r(t)[1 - u(t-2)]$$

which can be simplified as in Method 2 to get the same result.

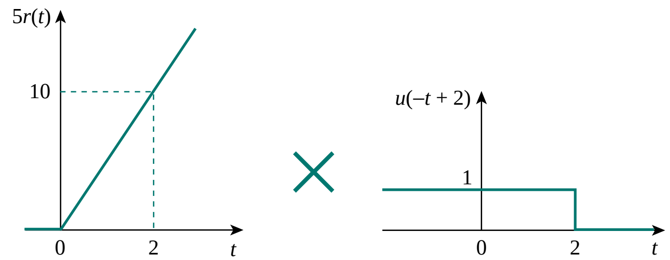


Figure 7.38 Decomposition of $v(t)$ in Fig. 7.35.

PRACTICE PROBLEM 7.7

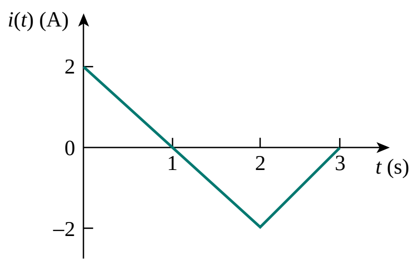


Figure 7.39 For Practice Prob. 7.7.

Refer to Fig. 7.39. Express $i(t)$ in terms of singularity functions.

Answer: $2u(t) - 2r(t) + 4r(t-2) - 2r(t-3)$.

EXAMPLE 7.8

Given the signal

$$g(t) = \begin{cases} 3, & t < 0 \\ -2, & 0 < t < 1 \\ 2t - 4, & t > 1 \end{cases}$$

express $g(t)$ in terms of step and ramp functions.

Solution:

The signal $g(t)$ may be regarded as the sum of three functions specified within the three intervals $t < 0$, $0 < t < 1$, and $t > 1$.

For $t < 0$, $g(t)$ may be regarded as 3 multiplied by $u(-t)$, where $u(-t) = 1$ for $t < 0$ and 0 for $t > 0$. Within the time interval $0 < t < 1$, the function may be considered as -2 multiplied by a gated function $[u(t) - u(t-1)]$. For $t > 1$, the function may be regarded as $2t - 4$ multiplied by the unit step function $u(t-1)$. Thus,

$$\begin{aligned} g(t) &= 3u(-t) - 2[u(t) - u(t-1)] + (2t-4)u(t-1) \\ &= 3u(-t) - 2u(t) + (2t - 4 + 2)u(t-1) \\ &= 3u(-t) - 2u(t) + 2(t-1)u(t-1) \\ &= 3u(-t) - 2u(t) + 2r(t-1) \end{aligned}$$

One may avoid the trouble of using $u(-t)$ by replacing it with $1 - u(t)$. Then

$$g(t) = 3[1 - u(t)] - 2u(t) + 2r(t-1) = 3 - 5u(t) + 2r(t-1)$$

Alternatively, we may plot $g(t)$ and apply Method 1 from Example 7.7.

PRACTICE PROBLEM 7.8

If

$$h(t) = \begin{cases} 0, & t < 0 \\ 4, & 0 < t < 2 \\ 6 - t, & 2 < t < 6 \\ 0, & t > 6 \end{cases}$$

express $h(t)$ in terms of the singularity functions.

Answer: $4u(t) - r(t-2) + r(t-6)$.

EXAMPLE 7.9

Evaluate the following integrals involving the impulse function:

$$\int_0^{10} (t^2 + 4t - 2)\delta(t-2)\,dt$$

$$\int_{-\infty}^{\infty} (\delta(t-1)e^{-t}\cos t + \delta(t+1)e^{-t}\sin t)dt$$

Solution:

For the first integral, we apply the sifting property in Eq. (7.32).

$$\int_0^{10} (t^2 + 4t - 2)\delta(t-2)dt = (t^2 + 4t - 2)|_{t=2} = 4 + 8 - 2 = 10$$

Similarly, for the second integral,

$$\begin{aligned}
&\int_{-\infty}^{\infty} (\delta(t-1)e^{-t}\cos t + \delta(t+1)e^{-t}\sin t)dt \\
&\quad = e^{-t}\cos t|_{t=1} + e^{-t}\sin t|_{t=-1} \\
&\quad = e^{-1}\cos 1 + e^{1}\sin(-1) = 0.1988 - 2.2873 = -2.0885
\end{aligned}$$

PRACTICE PROBLEM 7.9

Evaluate the following integrals:

$$\int_{-\infty}^{\infty} (t^3 + 5t^2 + 10)\delta(t+3)\,dt, \qquad \int_0^{10} \delta(t-\pi)\cos 3t\,dt$$

Answer: 28, −1.

7.5 STEP RESPONSE OF AN RC CIRCUIT

Electronic Testing Tutorials

When the dc source of an RC circuit is suddenly applied, the voltage or current source can be modeled as a step function, and the response is known as a *step response.*

> The **step response** of a circuit is its behavior when the excitation is the step function, which may be a voltage or a current source.

The step response is the response of the circuit due to a sudden application of a dc voltage or current source.

Consider the RC circuit in Fig. 7.40(a) which can be replaced by the circuit in Fig. 7.40(b), where V_s is a constant, dc voltage source. Again, we select the capacitor voltage as the circuit response to be determined. We assume an initial voltage V_0 on the capacitor, although this is not necessary for the step response. Since the voltage of a capacitor cannot change instantaneously,

$$v(0^-) = v(0^+) = V_0 \tag{7.40}$$

where $v(0^-)$ is the voltage across the capacitor just before switching and $v(0^+)$ is its voltage immediately after switching. Applying KCL, we have

$$C\frac{dv}{dt} + \frac{v - V_s u(t)}{R} = 0$$

or

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC}u(t) \tag{7.41}$$

where v is the voltage across the capacitor. For $t > 0$, Eq. (7.41) becomes

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC} \tag{7.42}$$

Rearranging terms gives

$$\frac{dv}{dt} = -\frac{v - V_s}{RC}$$

or

$$\frac{dv}{v - V_s} = -\frac{dt}{RC} \tag{7.43}$$

Integrating both sides and introducing the initial conditions,

$$\ln(v - V_s)\Big|_{V_0}^{v(t)} = -\frac{t}{RC}\Big|_0^t$$

$$\ln(v(t) - V_s) - \ln(V_0 - V_s) = -\frac{t}{RC} + 0$$

or

$$\ln\frac{v - V_s}{V_0 - V_s} = -\frac{t}{RC} \tag{7.44}$$

Taking the exponential of both sides

$$\frac{v - V_s}{V_0 - V_s} = e^{-t/\tau}, \qquad \tau = RC$$

$$v - V_s = (V_0 - V_s)e^{-t/\tau}$$

or

$$v(t) = V_s + (V_0 - V_s)e^{-t/\tau}, \qquad t > 0 \tag{7.45}$$

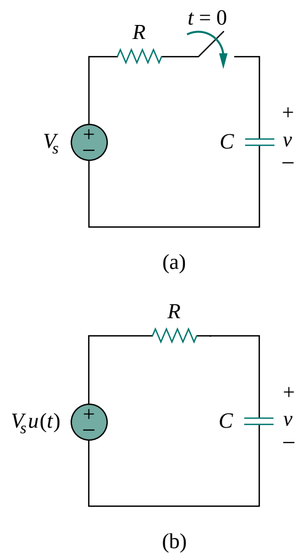


Figure 7.40 An RC circuit with voltage step input.

Thus,

$$\boxed{v(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)e^{-t/\tau}, & t > 0 \end{cases}} \tag{7.46}$$

This is known as the *complete response* of the RC circuit to a sudden application of a dc voltage source, assuming the capacitor is initially charged. The reason for the term "complete" will become evident a little later. Assuming that $V_s > V_0$, a plot of $v(t)$ is shown in Fig. 7.41.

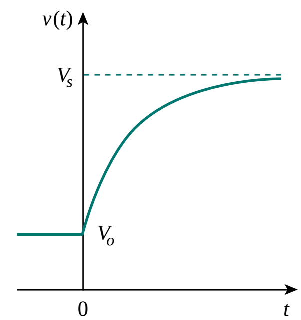


Figure 7.41 Response of an RC circuit with initially charged capacitor.

If we assume that the capacitor is uncharged initially, we set $V_0 = 0$ in Eq. (7.46) so that

$$v(t) = \begin{cases} 0, & t < 0 \\ V_s(1 - e^{-t/\tau}), & t > 0 \end{cases} \tag{7.47}$$

which can be written alternatively as

$$v(t) = V_s(1 - e^{-t/\tau})u(t) \tag{7.48}$$

This is the complete step response of the RC circuit when the capacitor is initially uncharged. The current through the capacitor is obtained from Eq. (7.47) using $i(t) = C\,dv/dt$. We get

$$i(t) = C\frac{dv}{dt} = \frac{C}{\tau}V_s e^{-t/\tau}, \qquad \tau = RC, \quad t > 0$$

or

$$i(t) = \frac{V_s}{R}e^{-t/\tau}u(t) \tag{7.49}$$

Figure 7.42 shows the plots of capacitor voltage $v(t)$ and capacitor current $i(t)$.

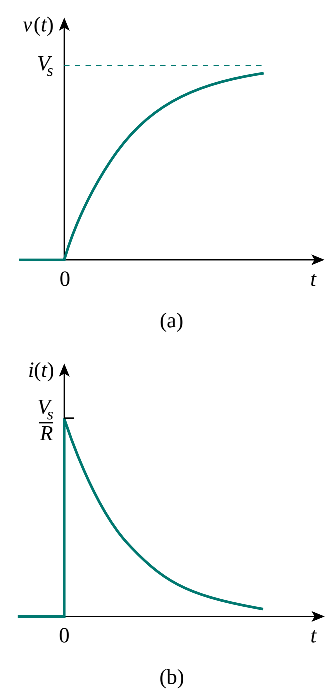


Figure 7.42 Step response of an RC circuit with initially uncharged capacitor: (a) voltage response, (b) current response.

Rather than going through the derivations above, there is a systematic approach—or rather, a short-cut method—for finding the step response of an RC or RL circuit. Let us reexamine Eq. (7.45), which is more general than Eq. (7.48). It is evident that $v(t)$ has two components. Thus, we may write

$$v = v_f + v_n \tag{7.50}$$

where

$$v_f = V_s \tag{7.51}$$

and

$$v_n = (V_0 - V_s)e^{-t/\tau} \tag{7.52}$$

We know that v_n is the natural response of the circuit, as discussed in Section 7.2. Since this part of the response will decay to almost zero after five time constants, it is also called the *transient* response because it is a temporary response that will die out with time. Now, v_f is known as the *forced* response because it is produced by the circuit when an external "force" is applied (a voltage source in this case). It represents what the circuit is forced to do by the input excitation. It is also known as the *steady-state response*, because it remains a long time after the circuit is excited.

The **natural response** or **transient response** is the circuit's temporary response that will die out with time.

The **forced response** or **steady-state response** is the behavior of the circuit a long time after an external excitation is applied.

This is the same as saying that the complete response is the sum of the transient response and the steady-state response.

The complete response of the circuit is the sum of the natural response and the forced response. Therefore, we may write Eq. (7.45) as

$$\boxed{v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}} \tag{7.53}$$

where $v(0)$ is the initial voltage at $t = 0^+$ and $v(\infty)$ is the final or steady-state value. Thus, to find the step response of an RC circuit requires three things:

1. The initial capacitor voltage $v(0)$.
2. The final capacitor voltage $v(\infty)$.
3. The time constant τ.

Once we know $x(0)$, $x(\infty)$, and τ, almost all the circuit problems in this chapter can be solved using the formula

$$x(t) = x(\infty) + [x(0) - x(\infty)]\, e^{-t/\tau}$$

We obtain item 1 from the given circuit for $t < 0$ and items 2 and 3 from the circuit for $t > 0$. Once these items are determined, we obtain the response using Eq. (7.53). This technique equally applies to RL circuits, as we shall see in the next section.

Note that if the switch changes position at time $t = t_0$ instead of at $t = 0$, there is a time delay in the response so that Eq. (7.53) becomes

$$v(t) = v(\infty) + [v(t_0) - v(\infty)]e^{-(t-t_0)/\tau} \tag{7.54}$$

where $v(t_0)$ is the initial value at $t = t_0^+$. Keep in mind that Eq. (7.53) or (7.54) applies only to step responses, that is, when the input excitation is constant.

EXAMPLE 7.10

The switch in Fig. 7.43 has been in position A for a long time. At $t = 0$, the switch moves to B. Determine $v(t)$ for $t > 0$ and calculate its value at $t = 1$ s and 4 s.

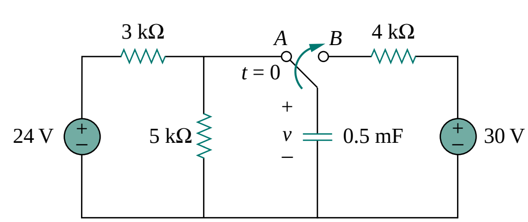


Figure 7.43 For Example 7.10.

Solution:

For $t < 0$, the switch is at position A. Since v is the same as the voltage across the 5-kΩ resistor, the voltage across the capacitor just before $t = 0$ is obtained by voltage division as

$$v(0^-) = \frac{5}{5+3}(24) = 15 \text{ V}$$

Using the fact that the capacitor voltage cannot change instantaneously,

$$v(0) = v(0^-) = v(0^+) = 15 \text{ V}$$

For $t > 0$, the switch is in position B. The Thevenin resistance connected to the capacitor is $R_{\text{Th}} = 4$ kΩ, and the time constant is

$$\tau = R_{\text{Th}}C = 4 \times 10^3 \times 0.5 \times 10^{-3} = 2 \text{ s}$$

Since the capacitor acts like an open circuit to dc at steady state, $v(\infty) = 30$ V. Thus,

$$\begin{aligned} v(t) &= v(\infty) + [v(0) - v(\infty)]e^{-t/\tau} \\ &= 30 + (15 - 30)e^{-t/2} = (30 - 15e^{-0.5t}) \text{ V} \end{aligned}$$

At $t = 1$,

$$v(1) = 30 - 15e^{-0.5} = 20.902 \text{ V}$$

At $t = 4$,

$$v(4) = 30 - 15e^{-2} = 27.97 \text{ V}$$

PRACTICE PROBLEM 7.10

Find $v(t)$ for $t > 0$ in the circuit in Fig. 7.44. Assume the switch has been open for a long time and is closed at $t = 0$. Calculate $v(t)$ at $t = 0.5$.

Answer: $-5 + 15e^{-2t}$ V, 0.5182 V.

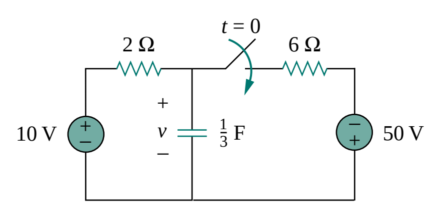


Figure 7.44 For Practice Prob. 7.10.

EXAMPLE 7.11

In Fig. 7.45, the switch has been closed for a long time and is opened at $t = 0$. Find i and v for all time.

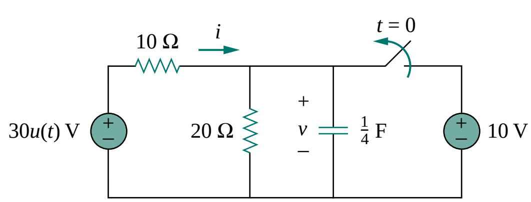


Figure 7.45 For Example 7.11.

Solution:

The resistor current i can be discontinuous at $t = 0$, while the capacitor voltage v cannot. Hence, it is always better to find v and then obtain i from v.

By definition of the unit step function,

$$30u(t) = \begin{cases} 0, & t < 0 \\ 30, & t > 0 \end{cases}$$

For $t < 0$, the switch is closed and $30u(t) = 0$, so that the $30u(t)$ voltage source is replaced by a short circuit and should be regarded as contributing nothing to v. Since the switch has been closed for a long time, the capacitor voltage has reached steady state and the capacitor acts like an open circuit. Hence, the circuit becomes that shown in Fig. 7.46(a) for $t < 0$. From this circuit we obtain

$$v = 10 \text{ V}, \qquad i = -\frac{v}{10} = -1 \text{ A}$$

Since the capacitor voltage cannot change instantaneously,

$$v(0) = v(0^-) = 10 \text{ V}$$

For $t > 0$, the switch is opened and the 10-V voltage source is disconnected from the circuit. The $30u(t)$ voltage source is now operative, so the circuit becomes that shown in Fig. 7.46(b). After a long time, the circuit reaches steady state and the capacitor acts like an open circuit again. We obtain $v(\infty)$ by using voltage division, writing

$$v(\infty) = \frac{20}{20 + 10}(30) = 20 \text{ V}$$

The Thevenin resistance at the capacitor terminals is

$$R_{\text{Th}} = 10 \parallel 20 = \frac{10 \times 20}{30} = \frac{20}{3}\ \Omega$$

and the time constant is

$$\tau = R_{\text{Th}}C = \frac{20}{3} \cdot \frac{1}{4} = \frac{5}{3} \text{ s}$$

Thus,

$$\begin{aligned} v(t) &= v(\infty) + [v(0) - v(\infty)]e^{-t/\tau} \\ &= 20 + (10 - 20)e^{-(3/5)t} = (20 - 10e^{-0.6t}) \text{ V} \end{aligned}$$

To obtain i, we notice from Fig. 7.46(b) that i is the sum of the currents through the 20-Ω resistor and the capacitor; that is,

$$\begin{aligned} i &= \frac{v}{20} + C\frac{dv}{dt} \\ &= 1 - 0.5e^{-0.6t} + 0.25(-0.6)(-10)e^{-0.6t} = (1 + e^{-0.6t}) \text{ A} \end{aligned}$$

Notice from Fig. 7.46(b) that $v + 10i = 30$ is satisfied, as expected. Hence,

$$v = \begin{cases} 10 \text{ V}, & t < 0 \\ (20 - 10e^{-0.6t}) \text{ V}, & t \geq 0 \end{cases}$$

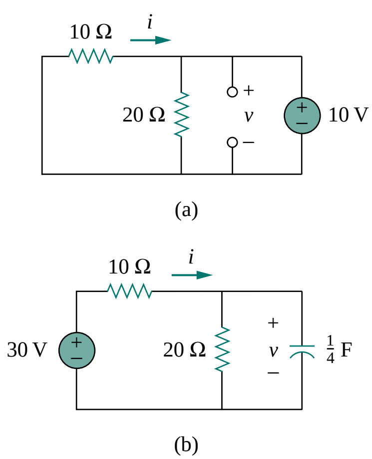


Figure 7.46 Solution of Example 7.11: (a) for $t < 0$, (b) for $t > 0$.

$$i = \begin{cases} -1\text{ A}, & t < 0 \\ (1 + e^{-0.6t})\text{ A}, & t > 0 \end{cases}$$

Notice that the capacitor voltage is continuous while the resistor current is not.

PRACTICE PROBLEM 7.11

The switch in Fig. 7.47 is closed at $t = 0$. Find $i(t)$ and $v(t)$ for all time. Note that $u(-t) = 1$ for $t < 0$ and 0 for $t > 0$. Also, $u(-t) = 1 - u(t)$.

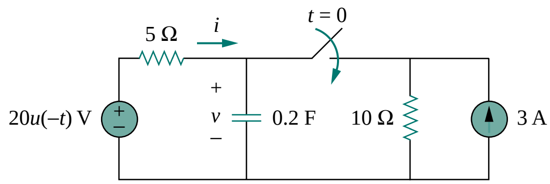


Figure 7.47 For Practice Prob. 7.11.

Answer: $i(t) = \begin{cases} 0, & t < 0 \\ -2(1 + e^{-1.5t})\text{ A}, & t > 0 \end{cases}$,

$$v = \begin{cases} 20\text{ V}, & t < 0 \\ 10(1 + e^{-1.5t})\text{ V}, & t > 0 \end{cases}$$

7.6 STEP RESPONSE OF AN *RL* CIRCUIT

Electronic Testing Tutorials

Consider the RL circuit in Fig. 7.48(a), which may be replaced by the circuit in Fig. 7.48(b). Again, our goal is to find the inductor current i as the circuit response. Rather than apply Kirchhoff's laws, we will use the simple technique in Eqs. (7.50) through (7.53). Let the response be the sum of the natural current and the forced current,

$$i = i_n + i_f \tag{7.55}$$

We know that the natural response is always a decaying exponential, that is,

$$i_n = Ae^{-t/\tau}, \qquad \tau = \frac{L}{R} \tag{7.56}$$

where A is a constant to be determined.

The forced response is the value of the current a long time after the switch in Fig. 7.48(a) is closed. We know that the natural response essentially dies out after five time constants. At that time, the inductor becomes a short circuit, and the voltage across it is zero. The entire source voltage V_s appears across R. Thus, the forced response is

$$i_f = \frac{V_s}{R} \tag{7.57}$$

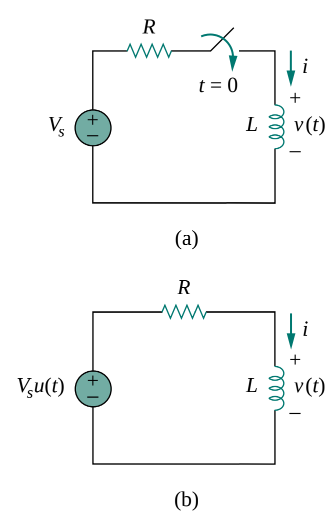


Figure 7.48 An RL circuit with a step input voltage.

Substituting Eqs. (7.56) and (7.57) into Eq. (7.55) gives

$$i = Ae^{-t/\tau} + \frac{V_s}{R} \tag{7.58}$$

We now determine the constant A from the initial value of i. Let I_0 be the initial current through the inductor, which may come from a source other than V_s. Since the current through the inductor cannot change instantaneously,

$$i(0^+) = i(0^-) = I_0 \tag{7.59}$$

Thus at $t = 0$, Eq. (7.58) becomes

$$I_0 = A + \frac{V_s}{R}$$

From this, we obtain A as

$$A = I_0 - \frac{V_s}{R}$$

Substituting for A in Eq. (7.58), we get

$$i(t) = \frac{V_s}{R} + \left(I_0 - \frac{V_s}{R}\right)e^{-t/\tau} \tag{7.60}$$

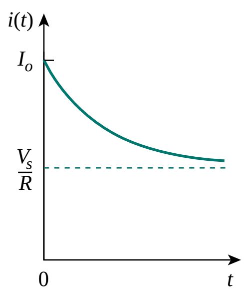


Figure 7.49 Total response of the *RL* circuit with initial inductor current I_0.

This is the complete response of the RL circuit. It is illustrated in Fig. 7.49. The response in Eq. (7.60) may be written as

$$\boxed{i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}} \tag{7.61}$$

where $i(0)$ and $i(\infty)$ are the initial and final values of i. Thus, to find the step response of an RL circuit requires three things:

1. The initial inductor current $i(0)$ at $t = 0^+$.
2. The final inductor current $i(\infty)$.
3. The time constant τ.

We obtain item 1 from the given circuit for $t < 0$ and items 2 and 3 from the circuit for $t > 0$. Once these items are determined, we obtain the response using Eq. (7.61). Keep in mind that this technique applies only for step responses.

Again, if the switching takes place at time $t = t_0$ instead of $t = 0$, Eq. (7.61) becomes

$$i(t) = i(\infty) + [i(t_0) - i(\infty)]e^{-(t-t_0)/\tau} \tag{7.62}$$

If $I_0 = 0$, then

$$i(t) = \begin{cases} 0, & t < 0 \\ \dfrac{V_s}{R}(1 - e^{-t/\tau}), & t > 0 \end{cases} \tag{7.63a}$$

or

$$i(t) = \frac{V_s}{R}(1 - e^{-t/\tau})u(t) \tag{7.63b}$$

This is the step response of the RL circuit. The voltage across the inductor is obtained from Eq. (7.63) using $v = L\,di/dt$. We get

$$v(t) = L\frac{di}{dt} = V_s \frac{L}{\tau R} e^{-t/\tau}, \qquad \tau = \frac{L}{R}, \quad t > 0$$

or

$$v(t) = V_s e^{-t/\tau} u(t) \tag{7.64}$$

Figure 7.50 shows the step responses in Eqs. (7.63) and (7.64).

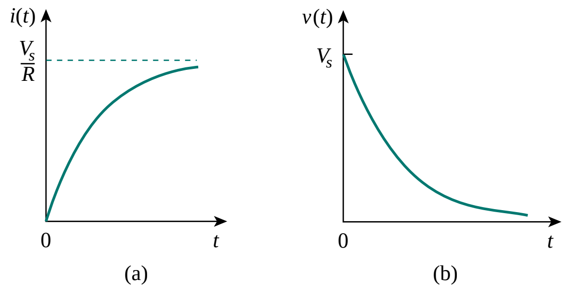


Figure 7.50 Step responses of an *RL* circuit with no initial inductor current: (a) current response, (b) voltage response.

EXAMPLE 7.12

Find $i(t)$ in the circuit in Fig. 7.51 for $t > 0$. Assume that the switch has been closed for a long time.

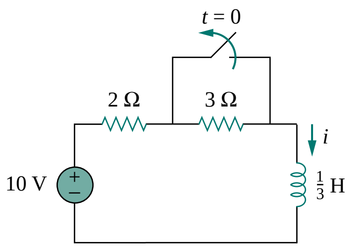


Figure 7.51 For Example 7.12.

Solution:

When $t < 0$, the 3-Ω resistor is short-circuited, and the inductor acts like a short circuit. The current through the inductor at $t = 0^-$ (i.e., just before $t = 0$) is

$$i(0^-) = \frac{10}{2} = 5 \text{ A}$$

Since the inductor current cannot change instantaneously,

$$i(0) = i(0^+) = i(0^-) = 5 \text{ A}$$

When $t > 0$, the switch is open. The 2-Ω and 3-Ω resistors are in series, so that

$$i(\infty) = \frac{10}{2+3} = 2 \text{ A}$$

The Thevenin resistance across the inductor terminals is

$$R_{\text{Th}} = 2 + 3 = 5\ \Omega$$

For the time constant,

$$\tau = \frac{L}{R_{\text{Th}}} = \frac{\frac{1}{3}}{5} = \frac{1}{15} \text{ s}$$

Thus,

$$\begin{aligned} i(t) &= i(\infty) + [i(0) - i(\infty)]e^{-t/\tau} \\ &= 2 + (5-2)e^{-15t} = 2 + 3e^{-15t} \text{ A}, \qquad t > 0 \end{aligned}$$

Check: In Fig. 7.51, for $t > 0$, KVL must be satisfied; that is,

$$10 = 5i + L\frac{di}{dt}$$

$$5i + L\frac{di}{dt} = [10 + 15e^{-15t}] + \left[\frac{1}{3}(3)(-15)e^{-15t}\right] = 10$$

This confirms the result.

PRACTICE PROBLEM 7.12

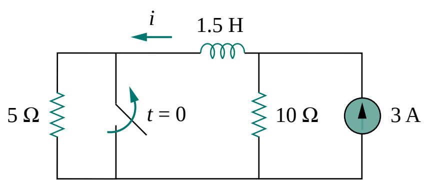


Figure 7.52 For Practice Prob. 7.12.

The switch in Fig. 7.52 has been closed for a long time. It opens at $t = 0$. Find $i(t)$ for $t > 0$.

Answer: $(2 + e^{-10t})$ A, $t > 0$.

EXAMPLE 7.13

At $t = 0$, switch 1 in Fig. 7.53 is closed, and switch 2 is closed 4 s later. Find $i(t)$ for $t > 0$. Calculate i for $t = 2$ s and $t = 5$ s.

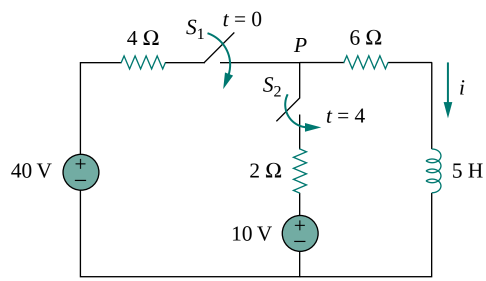


Figure 7.53 For Example 7.13.

Solution:

We need to consider the three time intervals $t \leq 0$, $0 \leq t \leq 4$, and $t \geq 4$ separately. For $t < 0$, switches S_1 and S_2 are open so that $i = 0$. Since the inductor current cannot change instantly,

$$i(0^-) = i(0) = i(0^+) = 0$$

For $0 \leq t \leq 4$, S_1 is closed so that the 4-Ω and 6-Ω resistors are in series. Hence, assuming for now that S_1 is closed forever,

$$i(\infty) = \frac{40}{4+6} = 4\text{ A}, \qquad R_{\text{Th}} = 4 + 6 = 10\ \Omega$$

$$\tau = \frac{L}{R_{\text{Th}}} = \frac{5}{10} = \frac{1}{2}\text{ s}$$

Thus,

$$\begin{aligned} i(t) &= i(\infty) + [i(0) - i(\infty)]e^{-t/\tau} \\ &= 4 + (0-4)e^{-2t} = 4(1 - e^{-2t}) \text{ A}, \qquad 0 \le t \le 4 \end{aligned}$$

For $t \ge 4$, S_2 is closed; the 10-V voltage source is connected, and the circuit changes. This sudden change does not affect the inductor current because the current cannot change abruptly. Thus, the initial current is

$$i(4) = i(4^-) = 4(1 - e^{-8}) \simeq 4 \text{ A}$$

To find $i(\infty)$, let v be the voltage at node P in Fig. 7.53. Using KCL,

$$\frac{40 - v}{4} + \frac{10 - v}{2} = \frac{v}{6} \qquad \Longrightarrow \qquad v = \frac{180}{11} \text{ V}$$

$$i(\infty) = \frac{v}{6} = \frac{30}{11} = 2.727 \text{ A}$$

The Thevenin resistance at the inductor terminals is

$$R_{\text{Th}} = 4 \parallel 2 + 6 = \frac{4 \times 2}{6} + 6 = \frac{22}{3} \ \Omega$$

and

$$\tau = \frac{L}{R_{\text{Th}}} = \frac{5}{\frac{22}{3}} = \frac{15}{22} \text{ s}$$

Hence,

$$i(t) = i(\infty) + [i(4) - i(\infty)]e^{-(t-4)/\tau}, \qquad t \ge 4$$

We need $(t-4)$ in the exponential because of the time delay. Thus,

$$\begin{aligned} i(t) &= 2.727 + (4 - 2.727)e^{-(t-4)/\tau}, \qquad \tau = \frac{15}{22} \\ &= 2.727 + 1.273e^{-1.4667(t-4)}, \qquad t \ge 4 \end{aligned}$$

Putting all this together,

$$i(t) = \begin{cases} 0, & t \le 0 \\ 4(1 - e^{-2t}), & 0 \le t \le 4 \\ 2.727 + 1.273e^{-1.4667(t-4)}, & t \ge 4 \end{cases}$$

At $t = 2$,

$$i(2) = 4(1 - e^{-4}) = 3.93 \text{ A}$$

At $t = 5$,

$$i(5) = 2.727 + 1.273e^{-1.4667} = 3.02 \text{ A}$$

PRACTICE PROBLEM 7.13

Switch S_1 in Fig. 7.54 is closed at $t = 0$, and switch S_2 is closed at $t = 2$ s. Calculate $i(t)$ for all t. Find $i(1)$ and $i(3)$.

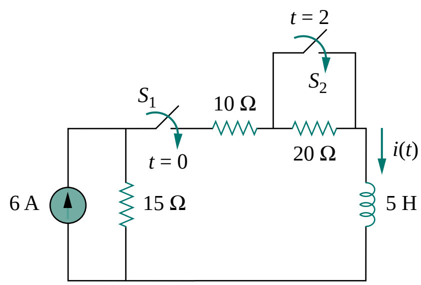


Figure 7.54 For Practice Prob. 7.13.

Answer:

$$i(t) = \begin{cases} 0, & t < 0 \\ 2(1 - e^{-9t}), & 0 < t < 2 \\ 3.6 - 1.6e^{-5(t-2)}, & t > 2 \end{cases}$$

$i(1) = 1.9997$ A, $i(3) = 3.589$ A.

Network Analysis

†7.7 FIRST-ORDER OP AMP CIRCUITS

An op amp circuit containing a storage element will exhibit first-order behavior. Differentiators and integrators treated in Section 6.6 are examples of first-order op amp circuits. Again, for practical reasons, inductors are hardly ever used in op amp circuits; therefore, the op amp circuits we consider here are of the RC type.

As usual, we analyze op amp circuits using nodal analysis. Sometimes, the Thevenin equivalent circuit is used to reduce the op amp circuit to one that we can easily handle. The following three examples illustrate the concepts. The first one deals with a source-free op amp circuit, while the other two involve step responses. The three examples have been carefully selected to cover all possible RC types of op amp circuits, depending on the location of the capacitor with respect to the op amp; that is, the capacitor can be located in the input, the output, or the feedback loop.

EXAMPLE 7.14

For the op amp circuit in Fig. 7.55(a), find v_o for $t > 0$, given that $v(0) = 3$ V. Let $R_f = 80$ kΩ, $R_1 = 20$ kΩ, and $C = 5\ \mu$F.

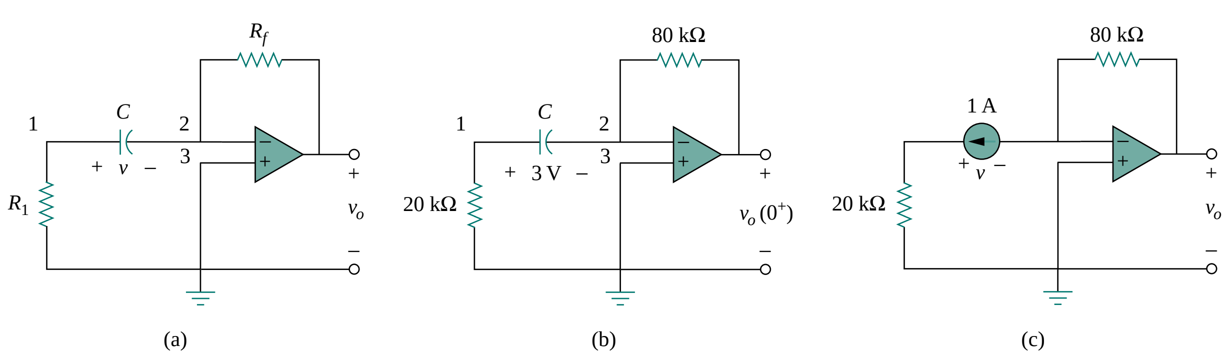


Figure 7.55 For Example 7.14.

Solution:

This problem can be solved in two ways:

METHOD 1 Consider the circuit in Fig. 7.55(a). Let us derive the appropriate differential equation using nodal analysis. If v_1 is the voltage at node 1, at that node, KCL gives

$$\frac{0 - v_1}{R_1} = C\frac{dv}{dt} \tag{7.14.1}$$

Since nodes 2 and 3 must be at the same potential, the potential at node 2 is zero. Thus, $v_1 - 0 = v$ or $v_1 = v$ and Eq. (7.14.1) becomes

$$\frac{dv}{dt} + \frac{v}{CR_1} = 0 \tag{7.14.2}$$

This is similar to Eq. (7.4b) so that the solution is obtained the same way as in Section 7.2, i.e.,

$$v(t) = V_0 e^{-t/\tau}, \qquad \tau = R_1 C \tag{7.14.3}$$

where V_0 is the initial voltage across the capacitor. But $v(0) = 3 = V_0$ and $\tau = 20 \times 10^3 \times 5 \times 10^{-6} = 0.1$. Hence,

$$v(t) = 3e^{-10t} \tag{7.14.4}$$

Applying KCL at node 2 gives

$$C\frac{dv}{dt} = \frac{0 - v_o}{R_f}$$

or

$$v_o = -R_f C\frac{dv}{dt} \tag{7.14.5}$$

Now we can find v_0 as

$$v_o = -80 \times 10^3 \times 5 \times 10^{-6}(-30e^{-10t}) = 12e^{-10t}\ \text{V}, \qquad t > 0$$

METHOD 2 Let us now apply the short-cut method from Eq. (7.53). We need to find $v_o(0^+)$, $v_o(\infty)$, and τ. Since $v(0^+) = v(0^-) = 3$ V, we apply KCL at node 2 in the circuit of Fig. 7.55(b) to obtain

$$\frac{3}{20{,}000} + \frac{0 - v_o(0^+)}{80{,}000} = 0$$

or $v_o(0^+) = 12$ V. Since the circuit is source free, $v(\infty) = 0$ V. To find τ, we need the equivalent resistance R_{eq} across the capacitor terminals. If we remove the capacitor and replace it by a 1-A current source, we have the circuit shown in Fig. 7.55(c). Applying KVL to the input loop yields

$$20{,}000(1) - v = 0 \qquad \Longrightarrow \qquad v = 20\ \text{kV}$$

Then

$$R_{\text{eq}} = \frac{v}{1} = 20\ \text{k}\Omega$$

and $\tau = R_{\text{eq}}C = 0.1$. Thus,

$$\begin{aligned} v_o(t) &= v_o(\infty) + [v_o(0) - v_o(\infty)]e^{-t/\tau} \\ &= 0 + (12 - 0)e^{-10t} = 12e^{-10t}\ \text{V}, \qquad t > 0 \end{aligned}$$

as before.

PRACTICE PROBLEM 7.14

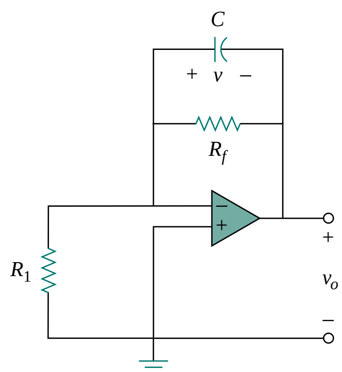


Figure 7.56 For Practice Prob. 7.14.

For the op amp circuit in Fig. 7.56, find v_o for $t > 0$ if $v(0) = 4$ V. Assume that $R_f = 50$ kΩ, $R_1 = 10$ kΩ, and $C = 10\ \mu$F.

Answer: $-4e^{-2t}$ V, $t > 0$.

EXAMPLE 7.15

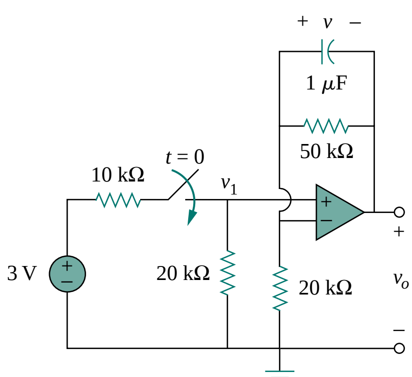


Figure 7.57 For Example 7.15.

Determine $v(t)$ and $v_o(t)$ in the circuit of Fig. 7.57.

Solution:

This problem can be solved in two ways, just like the previous example. However, we will apply only the second method. Since what we are looking for is the step response, we can apply Eq. (7.53) and write

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}, \qquad t > 0 \tag{7.15.1}$$

where we need only find the time constant τ, the initial value $v(0)$, and the final value $v(\infty)$. Notice that this applies strictly to the capacitor voltage due a step input. Since no current enters the input terminals of the op amp, the elements on the feedback loop of the op amp constitute an RC circuit, with

$$\tau = RC = 50 \times 10^3 \times 10^{-6} = 0.05 \tag{7.15.2}$$

For $t < 0$, the switch is open and there is no voltage across the capacitor. Hence, $v(0) = 0$. For $t > 0$, we obtain the voltage at node 1 by voltage division as

$$v_1 = \frac{20}{20 + 10} 3 = 2 \text{ V} \tag{7.15.3}$$

Since there is no storage element in the input loop, v_1 remains constant for all t. At steady state, the capacitor acts like an open circuit so that the op amp circuit is a noninverting amplifier. Thus,

$$v_o(\infty) = \left(1 + \frac{50}{20}\right) v_1 = 3.5 \times 2 = 7 \text{ V} \tag{7.15.4}$$

But

$$v_1 - v_o = v \tag{7.15.5}$$

so that

$$v(\infty) = 2 - 7 = -5 \text{ V}$$

Substituting τ, $v(0)$, and $v(\infty)$ into Eq. (7.15.1) gives

$$v(t) = -5 + [0 - (-5)]e^{-20t} = 5(e^{-20t} - 1) \text{ V}, \qquad t > 0 \qquad (7.15.6)$$

From Eqs. (7.15.3), (7.15.5), and (7.15.6), we obtain

$$v_o(t) = v_1(t) - v(t) = 7 - 5e^{-20t} \text{ V}, \qquad t > 0 \qquad (7.15.7)$$

PRACTICE PROBLEM 7.15

Find $v(t)$ and $v_o(t)$ in the op amp circuit of Fig. 7.58.

Answer: $40(1 - e^{-10t})$ mV, $40(e^{-10t} - 1)$ mV.

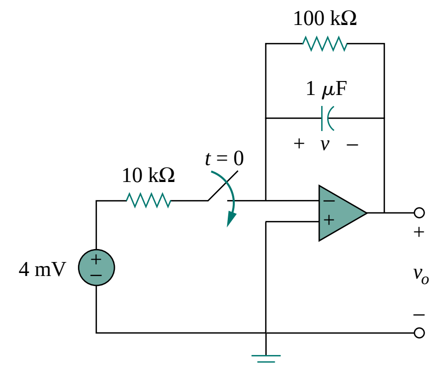


Figure 7.58 For Practice Prob. 7.15.

EXAMPLE 7.16

Find the step response $v_o(t)$ for $t > 0$ in the op amp circuit of Fig. 7.59. Let $v_i = 2u(t)$ V, $R_1 = 20$ kΩ, $R_f = 50$ kΩ, $R_2 = R_3 = 10$ kΩ, $C = 2\ \mu$F.

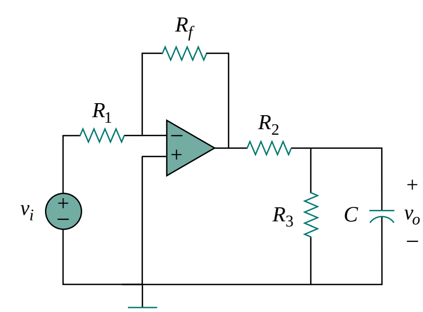


Figure 7.59 For Example 7.16.

Solution:

Notice that the capacitor in Example 7.14 is located in the input loop, while the capacitor in Example 7.15 is located in the feedback loop. In this example, the capacitor is located in the output of the op amp. Again, we can solve this problem directly using nodal analysis. However, using the Thevenin equivalent circuit may simplify the problem.

We temporarily remove the capacitor and find the Thevenin equivalent at its terminals. To obtain V_{Th}, consider the circuit in Fig. 7.60(a). Since the circuit is an inverting amplifier,

$$V_{ab} = -\frac{R_f}{R_1} v_i$$

By voltage division,

$$V_{\text{Th}} = \frac{R_3}{R_2 + R_3} V_{ab} = -\frac{R_3}{R_2 + R_3} \frac{R_f}{R_1} v_i$$

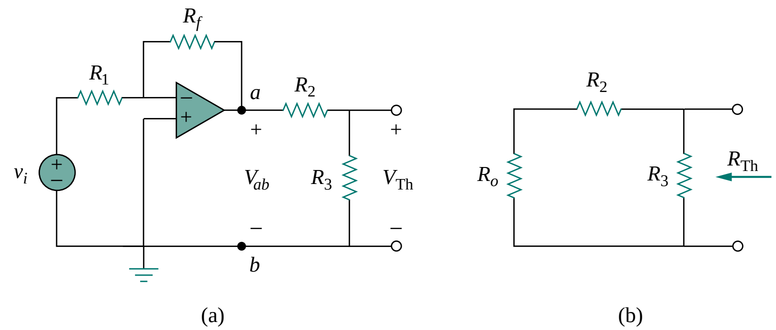


Figure 7.60 Obtaining V_{Th} and R_{Th} across the capacitor in Fig. 7.59.

To obtain R_{Th}, consider the circuit in Fig. 7.60(b), where R_o is the output resistance of the op amp. Since we are assuming an ideal op amp, $R_o = 0$, and

$$R_{Th} = R_2 \parallel R_3 = \frac{R_2 R_3}{R_2 + R_3}$$

Substituting the given numerical values,

$$V_{Th} = -\frac{R_3}{R_2 + R_3}\frac{R_f}{R_1}v_i = -\frac{10}{20}\frac{50}{20}2u(t) = -2.5u(t)$$

$$R_{Th} = \frac{R_2 R_3}{R_2 + R_3} = 5\ \text{k}\Omega$$

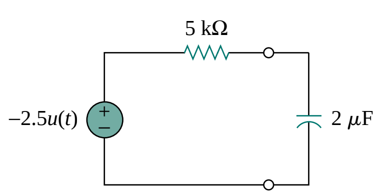


Figure 7.61 Thevenin equivalent circuit of the circuit in Fig. 7.59.

The Thevenin equivalent circuit is shown in Fig. 7.61, which is similar to Fig. 7.40. Hence, the solution is similar to that in Eq. (7.48); that is,

$$v_o(t) = -2.5(1 - e^{-t/\tau})\,u(t)$$

where $\tau = R_{Th}C = 5 \times 10^3 \times 2 \times 10^{-6} = 0.01$. Thus, the step response for $t > 0$ is

$$v_o(t) = 2.5(e^{-100t} - 1)\,u(t)\ \text{V}$$

PRACTICE PROBLEM 7.16

Obtain the step response $v_o(t)$ for the circuit of Fig. 7.62. Let $v_i = 2u(t)$ V, $R_1 = 20$ kΩ, $R_f = 40$ kΩ, $R_2 = R_3 = 10$ kΩ, $C = 2\ \mu$F.

Answer: $6(1 - e^{-50t})u(t)$ V.

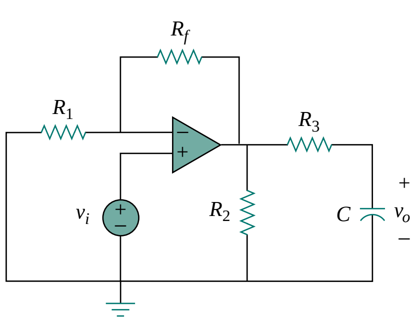


Figure 7.62 For Practice Prob. 7.16.

7.8 TRANSIENT ANALYSIS WITH *PSPICE*

As we discussed in Section 7.5, the transient response is the temporary response of the circuit that soon disappears. *PSpice* can be used to obtain the transient response of a circuit with storage elements. Section D.4 in Appendix D provides a review of transient analysis using *PSpice for Windows*. It is recommended that you read Section D.4 before continuing with this section.

PSpice uses "transient" to mean "function of time." Therefore, the transient response in *PSpice* may not actually die out as expected.

If necessary, dc *PSpice* analysis is first carried out to determine the initial conditions. Then the initial conditions are used in the transient *PSpice* analysis to obtain the transient responses. It is recommended but not necessary that during this dc analysis, all capacitors should be open-circuited while all inductors should be short-circuited.

EXAMPLE 7.17

Use *PSpice* to find the response $i(t)$ for $t > 0$ in the circuit of Fig. 7.63.

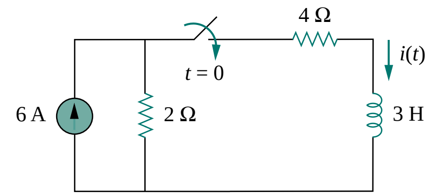


Figure 7.63 For Example 7.17.

Solution:

Solving this problem by hand gives $i(0) = 0$, $i(\infty) = 2$ A, $R_{\text{Th}} = 6$, $\tau = 3/6 = 0.5$ s, so that

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau} = 2(1 - e^{-2t}), \qquad t > 0$$

To use *PSpice*, we first draw the schematic as shown in Fig. 7.64. We recall from Appendix D that the part name for a close switch is Sw_tclose. We do not need to specify the initial condition of the inductor because *PSpice* will determine that from the circuit. By selecting **Analysis/Setup/Transient**, we set *Print Step* to 25 ms and *Final Step* to $5\tau = 2.5$ s. After saving the circuit, we simulate by selecting **Analysis/Simulate**. In the Probe menu, we select **Trace/Add** and display −I(L1) as the current through the inductor. Figure 7.65 shows the plot of $i(t)$, which agrees with that obtained by hand calculation.

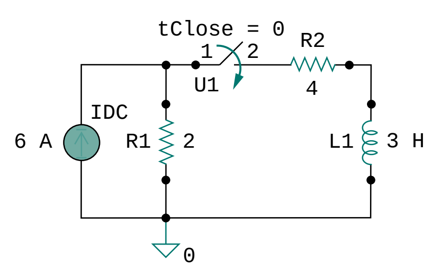


Figure 7.64 The schematic of the circuit in Fig. 7.63.

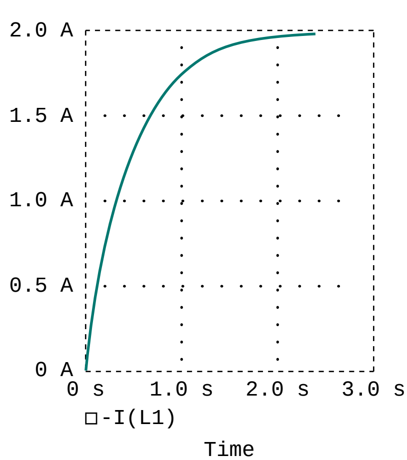


Figure 7.65 For Example 7.17; the response of the circuit in Fig. 7.63.

Note that the negative sign on I(L1) is needed because the current enters through the upper terminal of the inductor, which happens to be the negative terminal after one counterclockwise rotation. A way to avoid the negative sign is to ensure that current enters pin 1 of the inductor. To obtain this desired direction of positive current flow, the initially horizontal inductor symbol should be rotated counterclockwise 270° and placed in the desired location.

PRACTICE PROBLEM 7.17

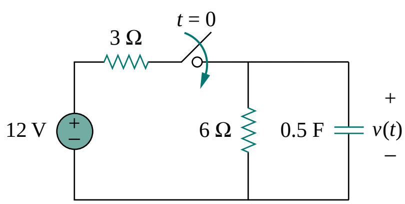


Figure 7.66 For Practice Prob. 7.17.

For the circuit in Fig. 7.66, use *PSpice* to find $v(t)$ for $t > 0$.

Answer: $v(t) = 8(1 - e^{-t})$ V, $t > 0$. The response is similar in shape to that in Fig. 7.65.

EXAMPLE 7.18

In the circuit in Fig. 7.67, determine the response $v(t)$.

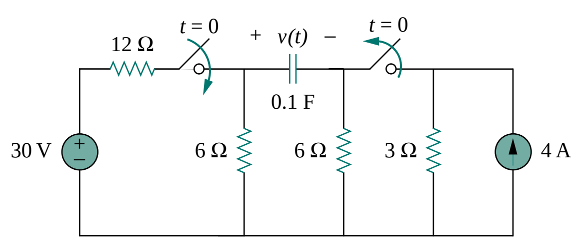


Figure 7.67 For Example 7.18.

Solution:

There are two ways of solving this problem using *PSpice.*

METHOD 1 One way is to first do the dc *PSpice* analysis to determine the initial capacitor voltage. The schematic of the revelant circuit is in Fig. 7.68(a). Two pseudocomponent VIEWPOINTs are inserted to measure the voltages at nodes 1 and 2. When the circuit is simulated, we obtain the displayed values in Fig. 7.68(a) as $V_1 = 0$ V and $V_2 = 8$ V. Thus the initial capacitor voltage is $v(0) = V_1 - V_2 = -8$ V. The *PSpice* transient analysis uses this value along with the schematic in Fig. 7.68(b). Once the circuit in Fig. 7.68(b) is drawn, we insert the capacitor initial voltage as IC $= -8$. We select **Analysis/Setup/Transient** and set *Print Step* to 0.1 s and *Final Step* to $4\tau = 4$ s. After saving the circuit, we select **Analysis/Simulate** to simulate the circuit. In the Probe menu, we select

Trace/Add and display V(R2:2) - V(R3:2) or V(C1:1) - V(C1:2) as the capacitor voltage $v(t)$. The plot of $v(t)$ is shown in Fig. 7.69. This agrees with the result obtained by hand calculation, $v(t) = 10 - 18e^{-t}$.

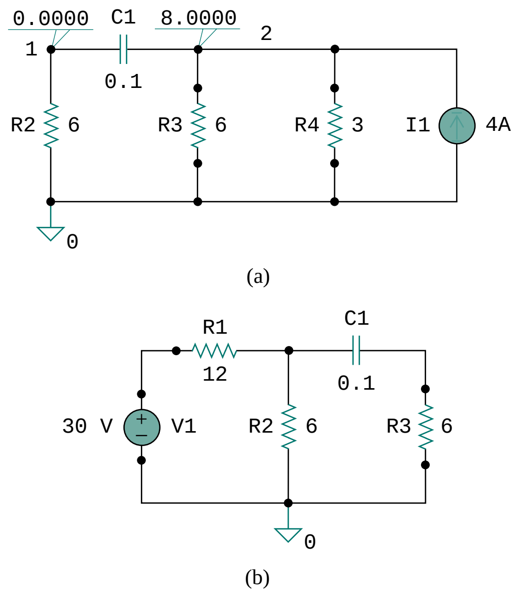


Figure 7.68 (a) Schematic for dc analysis to get $v(0)$, (b) schematic for transient analysis used in getting the response $v(t)$.

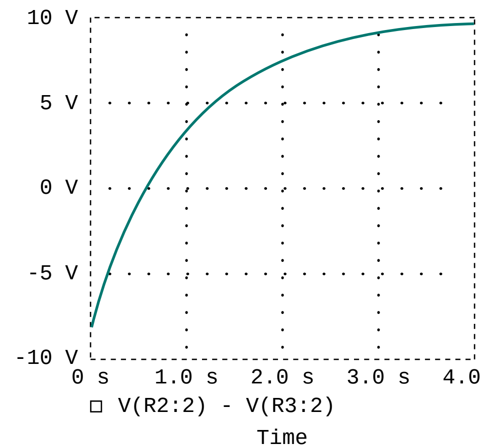


Figure 7.69 Response $v(t)$ for the circuit in Fig. 7.67.

METHOD 2 We can simulate the circuit in Fig. 7.67 directly, since *PSpice* can handle the open and close switches and determine the initial conditions automatically. Using this approach, the schematic is drawn as shown in Fig. 7.70. After drawing the circuit, we select **Analysis/Setup/Transient** and set *Print Step* to 0.1 s and *Final Step* to $4\tau = 4$ s. We save the circuit, then select **Analysis/Simulate** to simulate the circuit. In the Probe menu, we select **Trace/Add** and display V(R2:2) - V(R3:2) as the capacitor voltage $v(t)$. The plot of $v(t)$ is the same as that shown in Fig. 7.69.

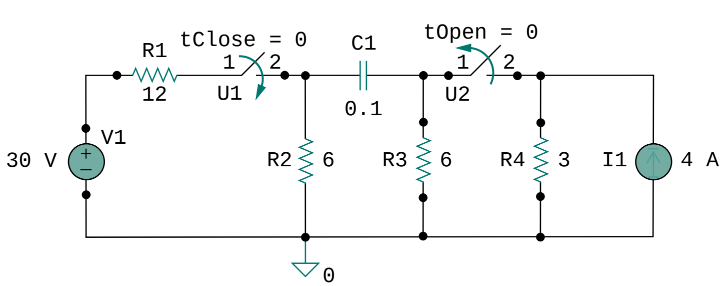


Figure 7.70 For Example 7.18.

PRACTICE PROBLEM 7.18

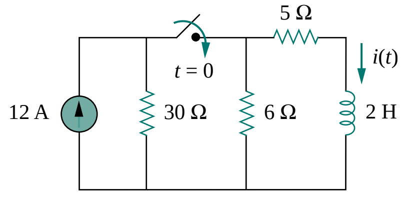


Figure 7.71 For Practice Prob. 7.18.

The switch in Fig. 7.71 was open for a long time but closed at $t = 0$. If $i(0) = 10$ A, find $i(t)$ for $t > 0$ by hand and also by *PSpice*.

Answer: $i(t) = 6 + 4e^{-5t}$ A. The plot of $i(t)$ obtained by *PSpice* analysis is shown in Fig. 7.72.

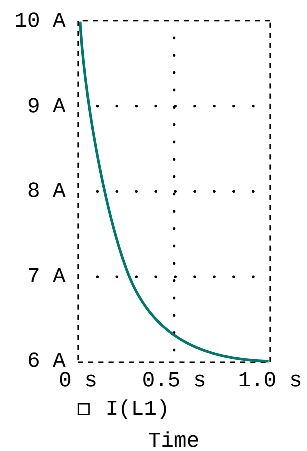


Figure 7.72 For Practice Prob. 7.18.

†7.9 APPLICATIONS

The various devices in which RC and RL circuits find applications include filtering in dc power supplies, smoothing circuits in digital communications, differentiators, integrators, delay circuits, and relay circuits. Some of these applications take advantage of the short or long time constants of the RC or RL circuits. We will consider four simple applications here. The first two are RC circuits, the last two are RL circuits.

7.9.1 Delay Circuits

An RC circuit can be used to provide various time delays. Figure 7.73 shows such a circuit. It basically consists of an RC circuit with the capacitor connected in parallel with a neon lamp. The voltage source can provide enough voltage to fire the lamp. When the switch is closed, the capacitor voltage increases gradually toward 110 V at a rate determined

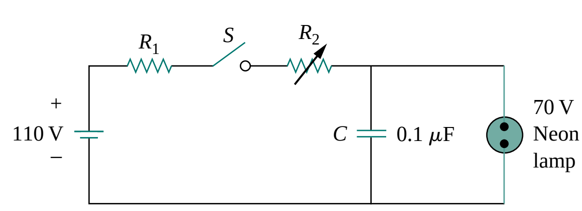


Figure 7.73 An RC delay circuit.

by the circuit's time constant, $(R_1 + R_2)C$. The lamp will act as an open circuit and not emit light until the voltage across it exceeds a particular level, say 70 V. When the voltage level is reached, the lamp fires (goes on), and the capacitor discharges through it. Due to the low resistance of the lamp when on, the capacitor voltage drops fast and the lamp turns off. The lamp acts again as an open circuit and the capacitor recharges. By adjusting R_2, we can introduce either short or long time delays into the circuit and make the lamp fire, recharge, and fire repeatedly every time constant $\tau = (R_1 + R_2)C$, because it takes a time period τ to get the capacitor voltage high enough to fire or low enough to turn off.

The warning blinkers commonly found on road construction sites are one example of the usefulness of such an *RC* delay circuit.

EXAMPLE 7.19

Consider the circuit in Fig. 7.73, and assume that $R_1 = 1.5\ \text{M}\Omega$, $0 < R < 2.5\ \text{M}\Omega$. (a) Calculate the extreme limits of the time constant of the circuit. (b) How long does it take for the lamp to glow for the first time after the switch is closed? Let R_2 assume its largest value.

Solution:

(a) The smallest value for R_2 is 0 Ω, and the corresponding time constant for the circuit is

$$\tau = (R_1 + R_2)C = (1.5 \times 10^6 + 0) \times 0.1 \times 10^{-6} = 0.15\ \text{s}$$

The largest value for R_2 is $2.5\ \text{M}\Omega$, and the corresponding time constant for the circuit is

$$\tau = (R_1 + R_2)C = (1.5 + 2.5) \times 10^6 \times 0.1 \times 10^{-6} = 0.4\ \text{s}$$

Thus, by proper circuit design, the time constant can be adjusted to introduce a proper time delay in the circuit.

(b) Assuming that the capacitor is initially uncharged, $v_C(0) = 0$, while $v_C(\infty) = 110$. But

$$v_C(t) = v_C(\infty) + [v_C(0) - v_C(\infty)]e^{-t/\tau} = 110[1 - e^{-t/\tau}]$$

where $\tau = 0.4$ s, as calculated in part (a). The lamp glows when $v_C = 70$ V. If $v_C(t) = 70$ V at $t = t_0$, then

$$70 = 110[1 - e^{-t_0/\tau}] \qquad \Longrightarrow \qquad \frac{7}{11} = 1 - e^{-t_0/\tau}$$

or

$$e^{-t_0/\tau} = \frac{4}{11} \qquad \Longrightarrow \qquad e^{t_0/\tau} = \frac{11}{4}$$

Taking the natural logarithm of both sides gives

$$t_0 = \tau \ln \frac{11}{4} = 0.4 \ln 2.75 = 0.4046\ \text{s}$$

A more general formula for finding t_0 is

$$t_0 = \tau \ln \frac{v(0) - v(\infty)}{v(t_0) - v(\infty)}$$

The lamp will fire repeatedly every τ seconds if and only if $t_0 < \tau$. In this example, that condition is not satisfied.

PRACTICE PROBLEM 7.19

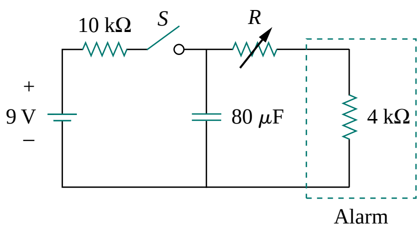


Figure 7.74 For Practice Prob. 7.19.

The RC circuit in Fig. 7.74 is designed to operate an alarm which activates when the current through it exceeds 120 μA. If $0 \le R \le 6$ kΩ, find the range of the time delay that the circuit can cause.

Answer: Between 47.23 ms and 124 ms.

7.9.2 Photoflash Unit

An electronic flash unit provides a common example of an RC circuit. This application exploits the ability of the capacitor to oppose any abrupt change in voltage. Figure 7.75 shows a simplified circuit. It consists essentially of a high-voltage dc supply, a current-limiting large resistor R_1, and a capacitor C in parallel with the flashlamp of low resistance R_2. When the switch is in position 1, the capacitor charges slowly due to the large time constant ($\tau_1 = R_1C$). As shown in Fig. 7.76, the capacitor voltage rises gradually from zero to V_s, while its current decreases gradually from $I_1 = V_s/R_1$ to zero. The charging time is approximately five times the time constant,

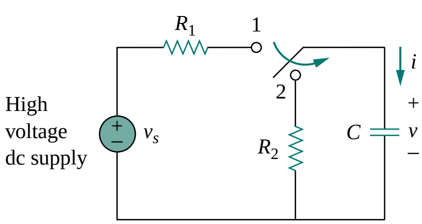


Figure 7.75 Circuit for a flash unit providing slow charge in position 1 and fast discharge in position 2.

$$t_{\text{charge}} = 5R_1C \tag{7.65}$$

With the switch in position 2, the capacitor voltage is discharged. The low resistance R_2 of the photolamp permits a high discharge current with peak $I_2 = V_s/R_2$ in a short duration, as depicted in Fig. 7.76(b). Discharging takes place in approximately five times the time constant,

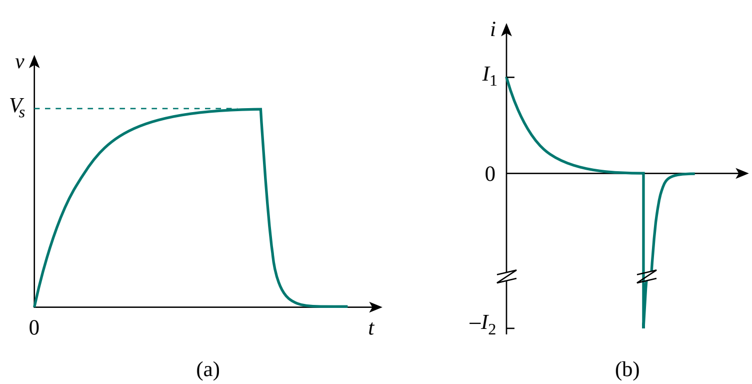


Figure 7.76 (a) Capacitor voltage showing slow charge and fast discharge, (b) capacitor current showing low charging current $I_1 = V_s/R_1$ and high discharge current $I_2 = V_s/R_2$.

$$t_{\text{discharge}} = 5R_2C \tag{7.66}$$

Thus, the simple RC circuit of Fig. 7.75 provides a short-duration, high-current pulse. Such a circuit also finds applications in electric spot welding and the radar transmitter tube.

EXAMPLE 7.20

An electronic flashgun has a current-limiting 6-kΩ resistor and 2000-μF electrolytic capacitor charged to 240 V. If the lamp resistance is 12 Ω, find: (a) the peak charging current, (b) the time required for the capacitor to fully charge, (c) the peak discharging current, (d) the total energy stored in the capacitor, and (e) the average power dissipated by the lamp.

Solution:

(a) The peak charging current is

$$I_1 = \frac{V_s}{R_1} = \frac{240}{6 \times 10^3} = 40 \text{ mA}$$

(b) From Eq. (7.65),

$$t_{\text{charge}} = 5R_1C = 5 \times 6 \times 10^3 \times 2000 \times 10^{-6} = 60 \text{ s} = 1 \text{ minute}$$

(c) The peak discharging current is

$$I_2 = \frac{V_s}{R_2} = \frac{240}{12} = 20 \text{ A}$$

(d) The energy stored is

$$W = \frac{1}{2}CV_s^2 = \frac{1}{2} \times 2000 \times 10^{-6} \times 240^2 = 57.6 \text{ J}$$

(e) The energy stored in the capacitor is dissipated across the lamp during the discharging period. From Eq. (7.66),

$$t_{\text{discharge}} = 5R_2C = 5 \times 12 \times 2000 \times 10^{-6} = 0.12 \text{ s}$$

Thus, the average power dissipated is

$$p = \frac{W}{t_{\text{discharge}}} = \frac{57.6}{0.12} = 480 \text{ W}$$

PRACTICE PROBLEM 7.20

The flash unit of a camera has a 2-mF capacitor charged to 80 V.

(a) How much charge is on the capacitor?

(b) What is the energy stored in the capacitor?

(c) If the flash fires in 0.8 ms, what is the average current through the flashtube?

(d) How much power is delivered to the flashtube?

(e) After a picture has been taken, the capacitor needs to be recharged by a power unit which supplies a maximum of 5 mA. How much time does it take to charge the capacitor?

Answer: (a) 0.16 C, (b) 6.4 J, (c) 200 A, (d) 8 kW, (e) 32 s.

7.9.3 Relay Circuits

A magnetically controlled switch is called a *relay*. A relay is essentially an electromagnetic device used to open or close a switch that controls another circuit. Figure 7.77(a) shows a typical relay circuit. The coil circuit is an RL circuit like that in Fig. 7.77(b), where R and L are the resistance and inductance of the coil. When switch S_1 in Fig. 7.77(a) is closed, the coil circuit is energized. The coil current gradually increases and produces a magnetic field. Eventually the magnetic field is sufficiently strong to pull the movable contact in the other circuit and close switch S_2. At this point, the relay is said to be *pulled in*. The time interval t_d between the closure of switches S_1 and S_2 is called the *relay delay time.*

Relays were used in the earliest digital circuits and are still used for switching high-power circuits.

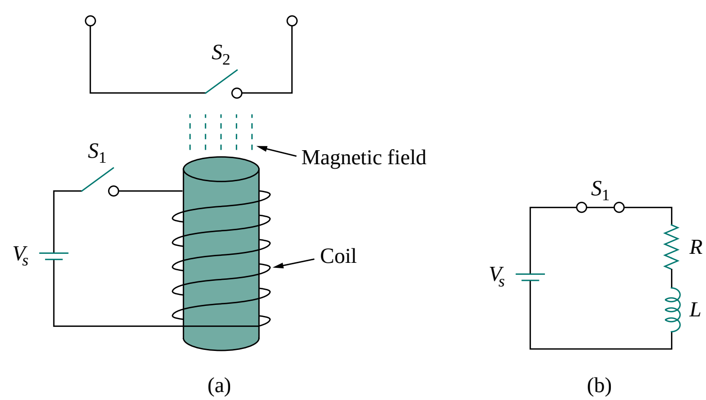


Figure 7.77 A relay circuit.

EXAMPLE 7.21

The coil of a certain relay is operated by a 12-V battery. If the coil has a resistance of 150 Ω and an inductance of 30 mH and the current needed to pull in is 50 mA, calculate the relay delay time.

Solution:

The current through the coil is given by

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$$

where

$$i(0) = 0, \qquad i(\infty) = \frac{12}{150} = 80 \text{ mA}$$

$$\tau = \frac{L}{R} = \frac{30 \times 10^{-3}}{150} = 0.2 \text{ ms}$$

Thus,

$$i(t) = 80[1 - e^{-t/\tau}] \text{ mA}$$

If $i(t_d) = 50$ mA, then

$$50 = 80[1 - e^{-t_d/\tau}] \qquad \Longrightarrow \qquad \frac{5}{8} = 1 - e^{-t_d/\tau}$$

or

$$e^{-t_d/\tau} = \frac{3}{8} \qquad \Longrightarrow \qquad e^{t_d/\tau} = \frac{8}{3}$$

By taking the natural logarithm of both sides, we get

$$t_d = \tau \ln \frac{8}{3} = 0.2 \ln \frac{8}{3} \text{ ms} = 0.1962 \text{ ms}$$

PRACTICE PROBLEM 7.21

A relay has a resistance of 200 Ω and an inductance of 500 mH. The relay contacts close when the current through the coil reaches 350 mA. What time elapses between the application of 110 V to the coil and contact closure?

Answer: 2.529 ms.

7.9.4 Automobile Ignition Circuit

The ability of inductors to oppose rapid change in current makes them useful for arc or spark generation. An automobile ignition system takes advantage of this feature.

The gasoline engine of an automobile requires that the fuel-air mixture in each cylinder be ignited at proper times. This is achieved by means of a spark plug (Fig. 7.78), which essentially consists of a pair of electrodes separated by an air gap. By creating a large voltage (thousands of volts) between the electrodes, a spark is formed across the air gap, thereby igniting the fuel. But how can such a large voltage be obtained from the car battery, which supplies only 12 V? This is achieved by means of an inductor (the spark coil) L. Since the voltage across the inductor is $v = L\,di/dt$, we can make di/dt large by creating a large change in current in a very short time. When the ignition switch in Fig. 7.78 is closed, the current through the inductor increases gradually and reaches the final value of $i = V_s/R$, where $V_s = 12$ V. Again, the time taken for the inductor to charge is five times the *time constant* of the circuit ($\tau = L/R$),

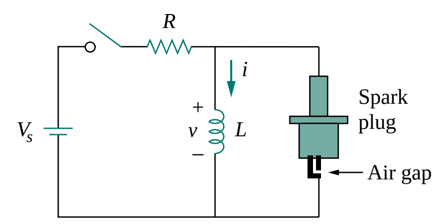


Figure 7.78 Circuit for an automobile ignition system.

$$t_{\text{charge}} = 5\frac{L}{R} \tag{7.67}$$

Since at steady state, i is constant, $di/dt = 0$ and the inductor voltage $v = 0$. When the switch suddenly opens, a large voltage is developed across the inductor (due to the rapidly collapsing field) causing a spark or arc in the air gap. The spark continues until the energy stored in the inductor is dissipated in the spark discharge. In laboratories, when one is working with inductive circuits, this same effect causes a very nasty shock, and one must exercise caution.

EXAMPLE 7.22

A solenoid with resistance 4 Ω and inductance 6 mH is used in an automobile ignition circuit similar to that in Fig. 7.78. If the battery supplies 12 V, determine: the final current through the solenoid when the switch is closed, the energy stored in the coil, and the voltage across the air gap, assuming that the switch takes 1 μs to open.

Solution:

The final current through the coil is

$$I = \frac{V_s}{R} = \frac{12}{4} = 3 \text{ A}$$

The energy stored in the coil is

$$W = \frac{1}{2}LI^2 = \frac{1}{2} \times 6 \times 10^{-3} \times 3^2 = 27 \text{ mJ}$$

The voltage across the gap is

$$V = L\frac{\Delta I}{\Delta t} = 6 \times 10^{-3} \times \frac{3}{1 \times 10^{-6}} = 18 \text{ kV}$$

PRACTICE PROBLEM 7.22

The spark coil of an automobile ignition system has a 20-mH inductance and a 5-Ω resistance. With a supply voltage of 12 V, calculate: the time needed for the coil to fully charge, the energy stored in the coil, and the voltage developed at the spark gap if the switch opens in 2 μs.

Answer: 20 ms, 57.6 mJ, and 24 kV.

7.10 SUMMARY

1. The analysis in this chapter is applicable to any circuit that can be reduced to an equivalent circuit comprising a resistor and a single energy-storage element (inductor or capacitor). Such a circuit is first-order because its behavior is described by a first-order differential equation. When analyzing RC and RL circuits, one must always keep in mind that the capacitor is an open circuit to steady-state dc conditions while the inductor is a short circuit to steady-state dc conditions.
2. The natural response is obtained when no independent source is present. It has the general form

   $$x(t) = x(0)e^{-t/\tau}$$

   where x represents current through (or voltage across) a resistor, a capacitor, or an inductor, and $x(0)$ is the initial value of x. The natural response is also called the *transient response* because it is the temporary response that vanishes with time.
3. The time constant τ is the time required for a response to decay to $1/e$ of its initial value. For RC circuits, $\tau = RC$ and for RL circuits, $\tau = L/R$.
4. The singularity functions include the unit step, the unit ramp function, and the unit impulse functions. The unit step function $u(t)$ is

   $$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

The unit impulse function is

$$\delta(t) = \begin{cases} 0, & t < 0 \\ \text{Undefined}, & t = 0 \\ 0, & t > 0 \end{cases}$$

The unit ramp function is

$$r(t) = \begin{cases} 0, & t \leq 0 \\ t, & t \geq 0 \end{cases}$$

5. The forced (or steady-state) response is the behavior of the circuit after an independent source has been applied for a long time.
6. The total or complete response consists of the natural response and the forced response.
7. The step response is the response of the circuit to a sudden application of a dc current or voltage. Finding the step response of a first-order circuit requires the initial value $x(0^+)$, the final value $x(\infty)$, and the time constant τ. With these three items, we obtain the step response as

$$x(t) = x(\infty) + [x(0^+) - x(\infty)]e^{-t/\tau}$$

A more general form of this equation is

$$x(t) = x(\infty) + [x(t_0^+) - x(\infty)]e^{-(t-t_0)/\tau}$$

Or we may write it as

$$\text{Instantaneous value} = \text{Final} + [\text{Initial} - \text{Final}]e^{-(t-t_0)/\tau}$$

8. *PSpice* is very useful for obtaining the transient response of a circuit.
9. Four practical applications of RC and RL circuits are: a delay circuit, a photoflash unit, a relay circuit, and an automobile ignition circuit.

REVIEW QUESTIONS

7.1 An RC circuit has $R = 2\ \Omega$ and $C = 4$ F. The time constant is:

(a) 0.5 s (b) 2 s (c) 4 s
(d) 8 s (e) 15 s

7.2 The time constant for an RL circuit with $R = 2\ \Omega$ and $L = 4$ H is:

(a) 0.5 s (b) 2 s (c) 4 s
(d) 8 s (e) 15 s

7.3 A capacitor in an RC circuit with $R = 2\ \Omega$ and $C = 4$ F is being charged. The time required for the capacitor voltage to reach 63.2 percent of its steady-state value is:

(a) 2 s (b) 4 s (c) 8 s
(d) 16 s (e) none of the above

7.4 An RL circuit has $R = 2\ \Omega$ and $L = 4$ H. The time needed for the inductor current to reach 40 percent of its steady-state value is:

(a) 0.5 s (b) 1 s (c) 2 s
(d) 4 s (e) none of the above

7.5 In the circuit of Fig. 7.79, the capacitor voltage just before $t = 0$ is:

(a) 10 V (b) 7 V (c) 6 V
(d) 4 V (e) 0 V

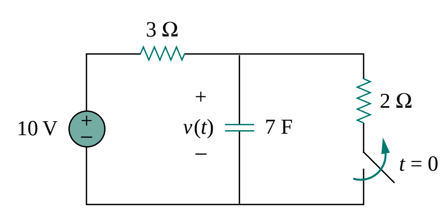


Figure 7.79 For Review Questions 7.5 and 7.6.

7.6 In the circuit of Fig. 7.79, $v(\infty)$ is:

(a) 10 V (b) 7 V (c) 6 V
(d) 4 V (e) 0 V

7.7 For the circuit of Fig. 7.80, the inductor current just before $t = 0$ is:

(a) 8 A (b) 6 A (c) 4 A
(d) 2 A (e) 0 A

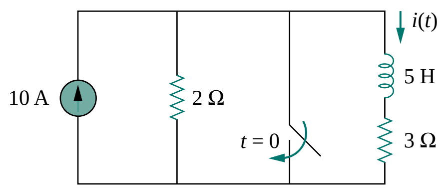


Figure 7.80 For Review Questions 7.7 and 7.8.

7.8 In the circuit of Fig. 7.80, $i(\infty)$ is:

(a) 8 A (b) 6 A (c) 4 A
(d) 2 A (e) 0 A

7.9 If v_s changes from 2 V to 4 V at $t = 0$, we may express v_s as:

(a) $\delta(t)$ V (b) $2u(t)$ V
(c) $2u(-t) + 4u(t)$ V (d) $2 + 2u(t)$ V
(e) $4u(t) - 2$ V

7.10 The pulse in Fig. 7.110(a) can be expressed in terms of singularity functions as:

(a) $2u(t) + 2u(t - 1)$ V (b) $2u(t) - 2u(t - 1)$ V
(c) $2u(t) - 4u(t - 1)$ V (d) $2u(t) + 4u(t - 1)$ V

Answers: 7.1d, 7.2b, 7.3c, 7.4b, 7.5d, 7.6a, 7.7c, 7.8e, 7.9c,d, 7.10b.

PROBLEMS

Section 7.2 The Source-Free *RC* Circuit

7.1 Show that Eq. (7.9) can be obtained by working with the current i in the RC circuit rather than working with the voltage v.

7.2 Find the time constant for the RC circuit in Fig. 7.81.

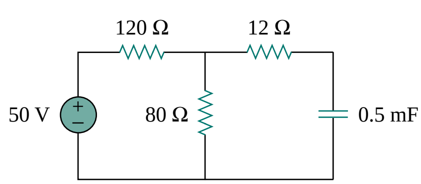


Figure 7.81 For Prob. 7.2.

7.3 Determine the time constant of the circuit in Fig. 7.82.

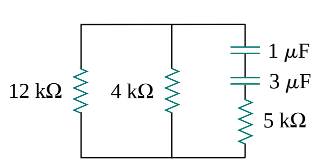


Figure 7.82 For Prob. 7.3.

7.4 Obtain the time constant of the circuit in Fig. 7.83.

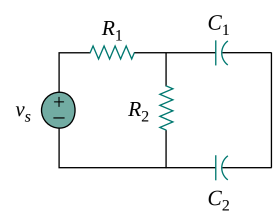


Figure 7.83 For Prob. 7.4.

7.5 The switch in Fig. 7.84 has been in position a for a long time, until $t = 4$ s when it is moved to position b and left there. Determine $v(t)$ at $t = 10$ s.

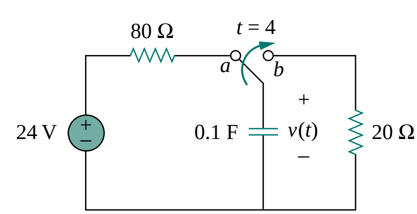


Figure 7.84 For Prob. 7.5.

7.6 If $v(0) = 20$ V in the circuit in Fig. 7.85, obtain $v(t)$ for $t > 0$.

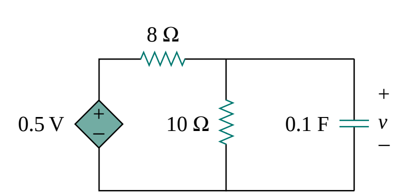


Figure 7.85 For Prob. 7.6.

7.7 For the circuit in Fig. 7.86, if

$$v = 10e^{-4t}\ \text{V} \qquad \text{and} \qquad i = 0.2e^{-4t}\ \text{A}, \qquad t > 0$$

(a) Find R and C.
(b) Determine the time constant.
(c) Calculate the initial energy in the capacitor.
(d) Obtain the time it takes to dissipate 50 percent of the initial energy.

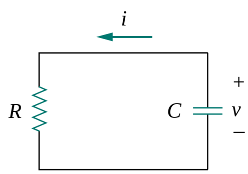


Figure 7.86 For Prob. 7.7.

7.8 In the circuit of Fig. 7.87, $v(0) = 20$ V. Find $v(t)$ for $t > 0$.

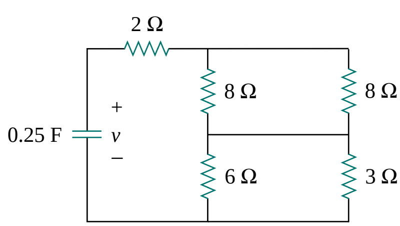


Figure 7.87 For Prob. 7.8.

7.9 Given that $i(0) = 3$ A, find $i(t)$ for $t > 0$ in the circuit in Fig. 7.88.

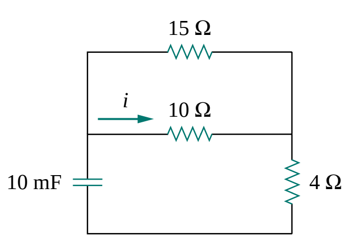


Figure 7.88 For Prob. 7.9.

Section 7.3 The Source-Free *RL* Circuit

7.10 Derive Eq. (7.20) by working with voltage v across the inductor of the RL circuit instead of working with the current i.

7.11 The switch in the circuit in Fig. 7.89 has been closed for a long time. At $t = 0$, the switch is opened. Calculate $i(t)$ for $t > 0$.

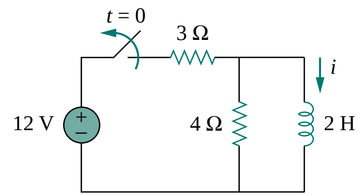


Figure 7.89 For Prob. 7.11.

7.12 For the circuit shown in Fig. 7.90, calculate the time constant.

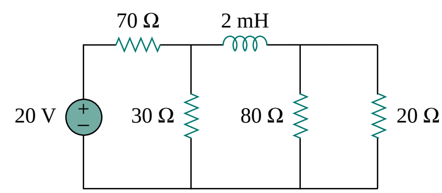


Figure 7.90 For Prob. 7.12.

7.13 What is the time constant of the circuit in Fig. 7.91?

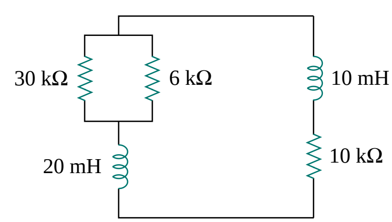


Figure 7.91 For Prob. 7.13.

7.14 Determine the time constant for each of the circuits in Fig. 7.92.

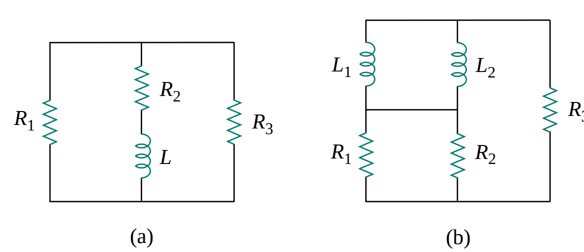


Figure 7.92 For Prob. 7.14.

7.15 Consider the circuit of Fig. 7.93. Find $v_o(t)$ if $i(0) = 2$ A and $v(t) = 0$.

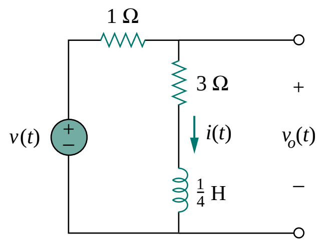


Figure 7.93 For Prob. 7.15.

7.16 For the circuit in Fig. 7.94, determine $v_o(t)$ when $i(0) = 1$ A and $v(t) = 0$.

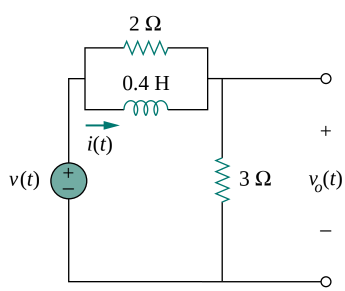


Figure 7.94 For Prob. 7.16.

7.17 In the circuit of Fig. 7.95, find $i(t)$ for $t > 0$ if $i(0) = 2$ A.

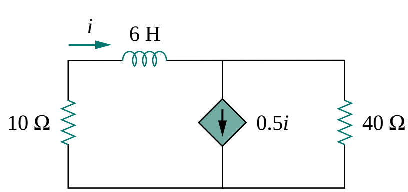


Figure 7.95 For Prob. 7.17.

7.18 For the circuit in Fig. 7.96,

$$v = 120e^{-50t} \text{ V}$$

and

$$i = 30e^{-50t} \text{ A}, \qquad t > 0$$

(a) Find L and R.

(b) Determine the time constant.

(c) Calculate the initial energy in the inductor.

(d) What fraction of the initial energy is dissipated in 10 ms?

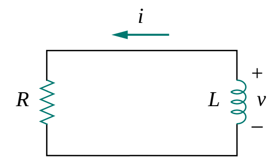


Figure 7.96 For Prob. 7.18.

7.19 In the circuit in Fig. 7.97, find the value of R for which energy stored in the inductor will be 1 J.

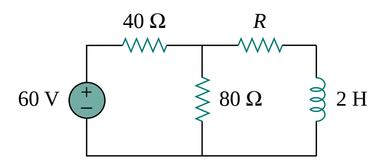


Figure 7.97 For Prob. 7.19.

7.20 Find $i(t)$ and $v(t)$ for $t > 0$ in the circuit of Fig. 7.98 if $i(0) = 10$ A.

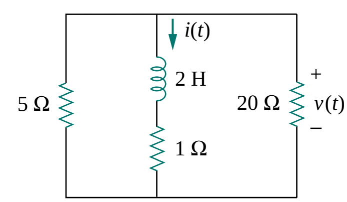


Figure 7.98 For Prob. 7.20.

7.21 Consider the circuit in Fig. 7.99. Given that $v_o(0) = 2$ V, find v_o and v_x for $t > 0$.

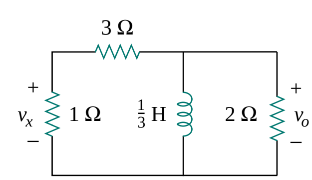


Figure 7.99 For Prob. 7.21.

Section 7.4 Singularity Functions

7.22 Express the following signals in terms of singularity functions.

(a) $v(t) = \begin{cases} 0, & t < 0 \\ -5, & t > 0 \end{cases}$

(b) $i(t) = \begin{cases} 0, & t < 1 \\ -10, & 1 < t < 3 \\ 10, & 3 < t < 5 \\ 0, & t > 5 \end{cases}$

(c) $x(t) = \begin{cases} t-1, & 1 < t < 2 \\ 1, & 2 < t < 3 \\ 4-t, & 3 < t < 4 \\ 0, & \text{Otherwise} \end{cases}$

(d) $y(t) = \begin{cases} 2, & t < 0 \\ -5, & 0 < t < 1 \\ 0, & t > 1 \end{cases}$

7.23 Express the signals in Fig. 7.100 in terms of singularity functions.

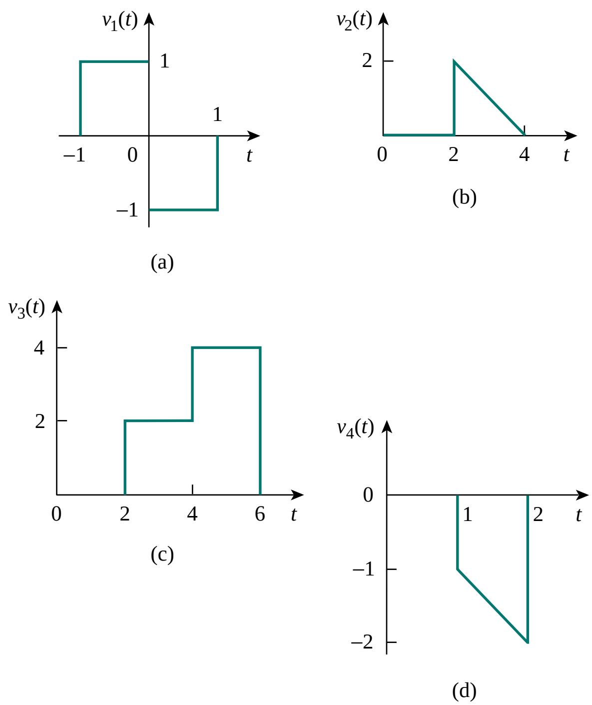


Figure 7.100 For Prob. 7.23.

7.24 Sketch the waveform that is represented by

$$v(t) = u(t) + u(t-1) - 3u(t-2) + 2u(t-3)$$

7.25 Sketch the waveform represented by

$$i(t) = r(t) + r(t-1) - u(t-2) - r(t-2) + r(t-3) + u(t-4)$$

7.26 Evaluate the following integrals involving the impulse functions:

(a) $\int_{-\infty}^{\infty} 4t^2 \delta(t-1)\, dt$

(b) $\int_{-\infty}^{\infty} 4t^2 \cos 2\pi t \delta(t-0.5)\, dt$

7.27 Evaluate the following integrals:

(a) $\int_{-\infty}^{\infty} e^{-4t^2} \delta(t-2)\, dt$

(b) $\int_{-\infty}^{\infty} [5\delta(t) + e^{-t}\delta(t) + \cos 2\pi t \delta(t)] dt$

7.28 The voltage across a 10-mH inductor is $20\delta(t-2)$ mV. Find the inductor current, assuming that the inductor is initially uncharged.

7.29 Find the solution of the following first-order differential equations subject to the specified initial conditions.

(a) $5\, dv/dt + 3v = 0, \quad v(0) = -2$

(b) $4\, dv/dt - 6v = 0, \quad v(0) = 5$

7.30 Solve for v in the following differential equations, subject to the stated initial condition.

(a) $dv/dt + v = u(t), \quad v(0) = 0$

(b) $2\, dv/dt - v = 3u(t), \quad v(0) = -6$

Section 7.5 Step Response of an *RC* Circuit

7.31 Calculate the capacitor voltage for $t < 0$ and $t > 0$ for each of the circuits in Fig. 7.101.

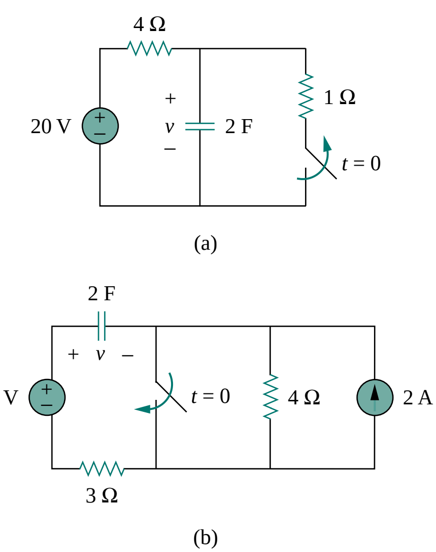


Figure 7.101 For Prob. 7.31.

7.32 Find the capacitor voltage for $t < 0$ and $t > 0$ for each of the circuits in Fig. 7.102.

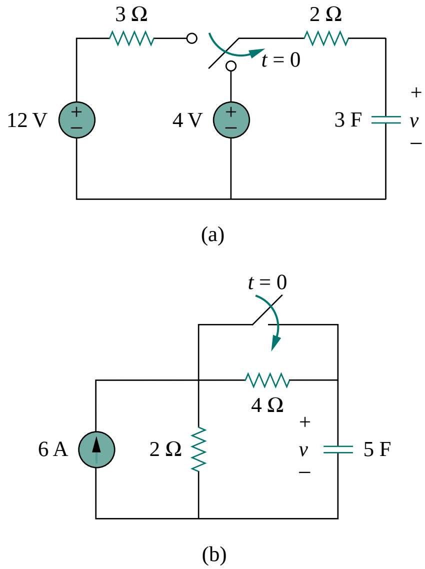


Figure 7.102 For Prob. 7.32.

7.33 For the circuit in Fig. 7.103, find $v(t)$ for $t > 0$.

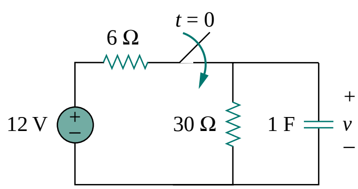


Figure 7.103 For Prob. 7.33.

7.34 (a) If the switch in Fig. 7.104 has been open for a long time and is closed at $t = 0$, find $v_o(t)$.

(b) Suppose that the switch has been closed for a long time and is opened at $t = 0$. Find $v_o(t)$.

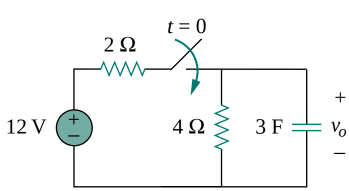


Figure 7.104 For Prob. 7.34.

7.35 Consider the circuit in Fig. 7.105. Find $i(t)$ for $t < 0$ and $t > 0$.

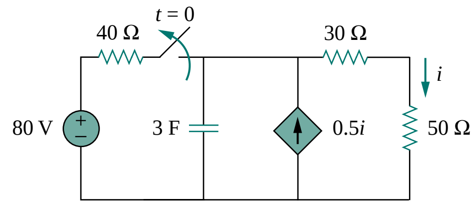


Figure 7.105 For Prob. 7.35.

7.36 The switch in Fig. 7.106 has been in position a for a long time. At $t = 0$, it moves to position b. Calculate $i(t)$ for all $t > 0$.

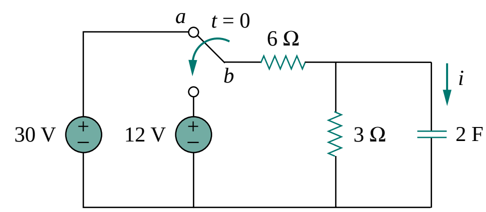


Figure 7.106 For Prob. 7.36.

7.37 Find the step responses $v(t)$ and $i(t)$ to $v_s = 5u(t)$ V in the circuit of Fig. 7.107.

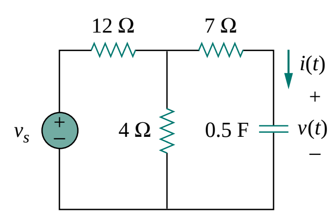


Figure 7.107 For Prob. 7.37.

7.38 Determine $v(t)$ for $t > 0$ in the circuit in Fig. 7.108 if $v(0) = 0$.

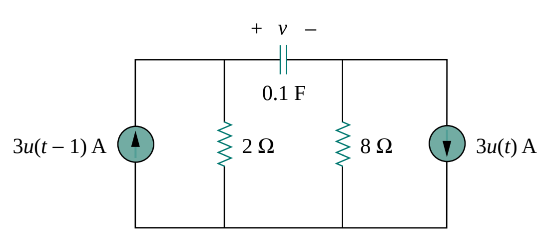


Figure 7.108 For Prob. 7.38.

7.39 Find $v(t)$ and $i(t)$ in the circuit of Fig. 7.109.

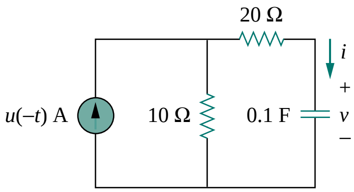
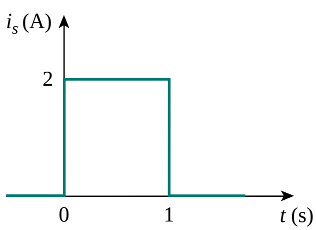
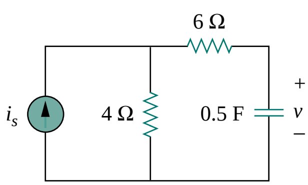


Figure 7.109 For Prob. 7.39.

7.40 If the waveform in Fig. 7.110(a) is applied to the circuit of Fig. 7.110(b), find $v(t)$. Assume $v(0) = 0$.



(a)



(b)

Figure 7.110 For Prob. 7.40 and Review Question 7.10.

***7.41** In the circuit in Fig. 7.111, find i_x for $t > 0$. Let $R_1 = R_2 = 1\ \text{k}\Omega$, $R_3 = 2\ \text{k}\Omega$, and $C = 0.25$ mF.

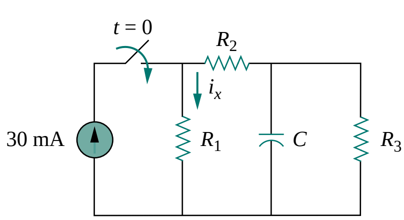


Figure 7.111 For Prob. 7.41.

Section 7.6 Step Response of an *RL* Circuit

7.42 Rather than applying the short-cut technique used in Section 7.6, use KVL to obtain Eq. (7.60).

7.43 For the circuit in Fig. 7.112, find $i(t)$ for $t > 0$.

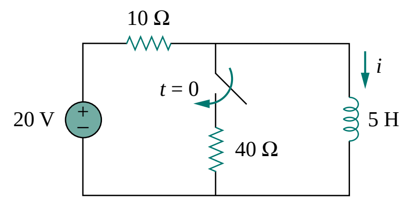
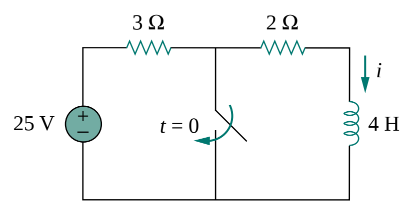
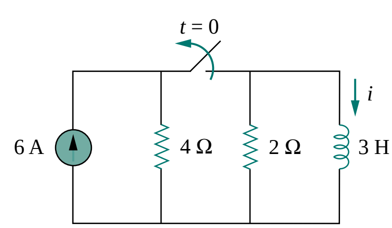


Figure 7.112 For Prob. 7.43.

7.44 Determine the inductor current $i(t)$ for both $t < 0$ and $t > 0$ for each of the circuits in Fig. 7.113.



(a)



(b)

Figure 7.113 For Prob. 7.44.

7.45 Obtain the inductor current for both $t < 0$ and $t > 0$ in each of the circuits in Fig. 7.114.

*An asterisk indicates a challenging problem.

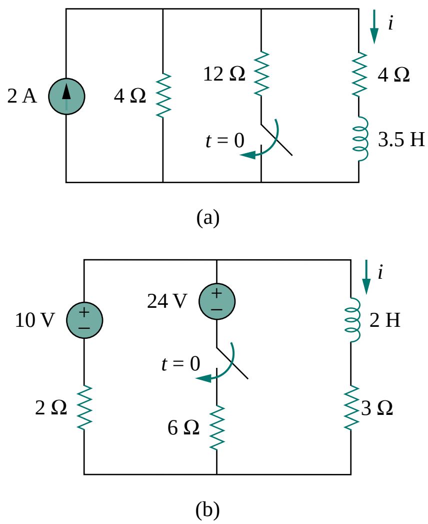


Figure 7.114 For Prob. 7.45.

7.46 Find $v(t)$ for $t < 0$ and $t > 0$ in the circuit in Fig. 7.115.

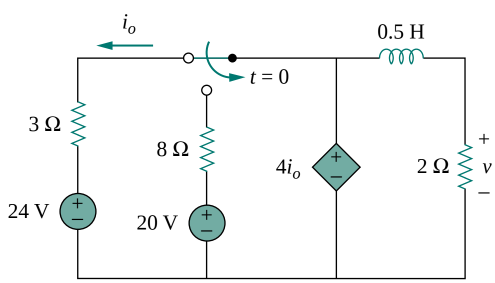


Figure 7.115 For Prob. 7.46.

7.47 For the network shown in Fig. 7.116, find $v(t)$ for $t > 0$.

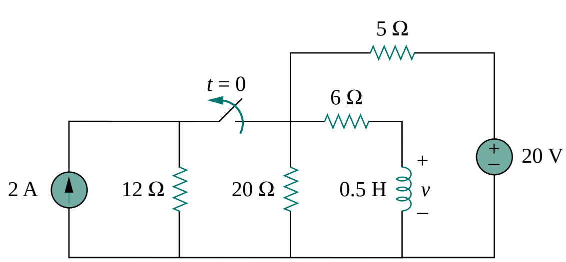


Figure 7.116 For Prob. 7.47.

*__7.48__ Find $i_1(t)$ and $i_2(t)$ for $t > 0$ in the circuit of Fig. 7.117.

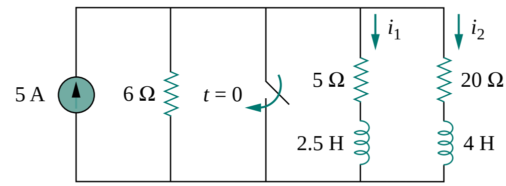


Figure 7.117 For Prob. 7.48.

7.49 Rework Prob. 7.15 if $i(0) = 10$ A and $v(t) = 20u(t)$ V.

7.50 Determine the step response $v_o(t)$ to $v_s = 18u(t)$ in the circuit of Fig. 7.118.

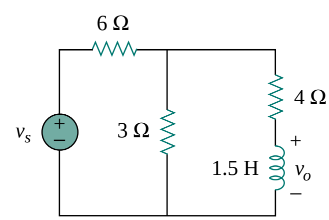


Figure 7.118 For Prob. 7.50.

7.51 Find $v(t)$ for $t > 0$ in the circuit of Fig. 7.119 if the initial current in the inductor is zero.

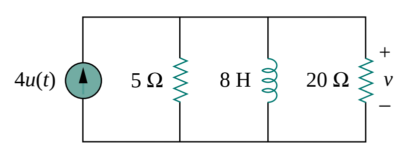


Figure 7.119 For Prob. 7.51.

7.52 In the circuit in Fig. 7.120, i_s changes from 5 A to 10 A at $t = 0$; that is, $i_s = 5u(-t) + 10u(t)$. Find v and i.

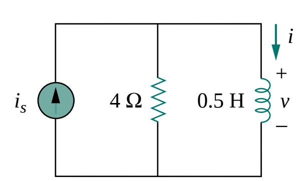


Figure 7.120 For Prob. 7.52.

7.53 For the circuit in Fig. 7.121, calculate $i(t)$ if $i(0) = 0$.

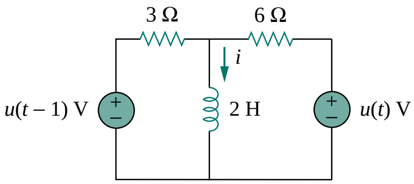


Figure 7.121 For Prob. 7.53.

7.54 Obtain $v(t)$ and $i(t)$ in the circuit of Fig. 7.122.

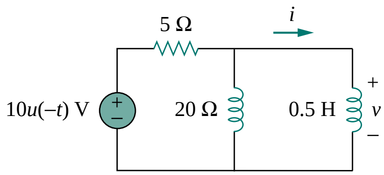


Figure 7.122 For Prob. 7.54.

7.55 Find $v_o(t)$ for $t > 0$ in the circuit of Fig. 7.123.

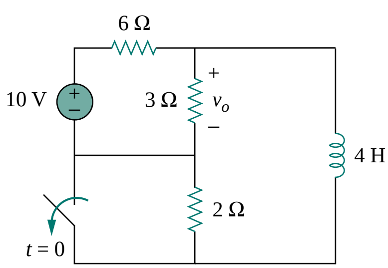


Figure 7.123 For Prob. 7.55.

7.56 If the input pulse in Fig. 7.124(a) is applied to the circuit in Fig. 7.124(b), determine the response $i(t)$.

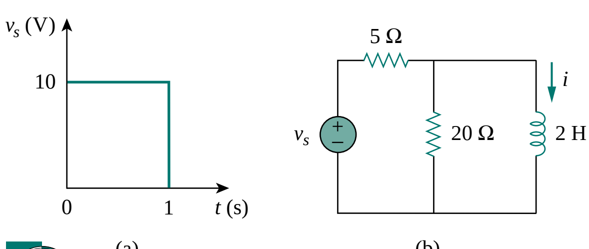


Figure 7.124 For Prob. 7.56.

Section 7.7 First-order Op Amp Circuits

7.57 Find the output current i_o for $t > 0$ in the op amp circuit of Fig. 7.125. Let $v(0) = -4$ V.

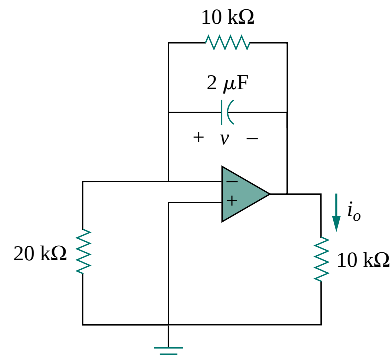


Figure 7.125 For Prob. 7.57.

7.58 If $v(0) = 5$ V, find $v_o(t)$ for $t > 0$ in the op amp circuit in Fig. 7.126. Let $R = 10$ kΩ and $C = 1$ μF.

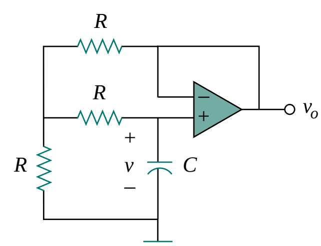


Figure 7.126 For Prob. 7.58.

7.59 Obtain v_o for $t > 0$ in the circuit of Fig. 7.127.

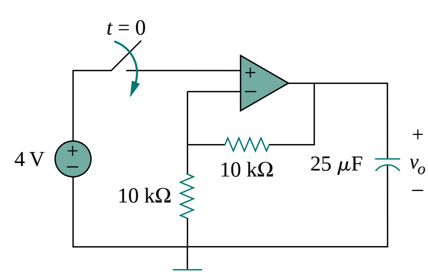


Figure 7.127 For Prob. 7.59.

7.60 For the op amp circuit in Fig. 7.128, find $v_o(t)$ for $t > 0$.

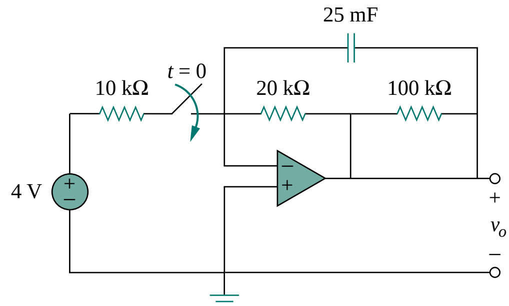


Figure 7.128 For Prob. 7.60.

7.61 Determine v_o for $t > 0$ when $v_s = 20$ mV in the op amp circuit of Fig. 7.129.

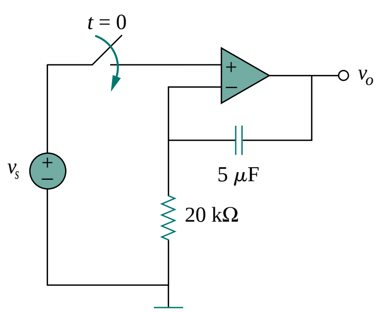


Figure 7.129 For Prob. 7.61.

7.62 For the op amp circuit in Fig. 7.130, find i_o for $t > 2$.

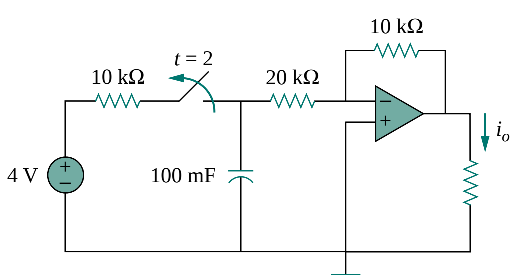


Figure 7.130 For Prob. 7.62.

7.63 Find i_o in the op amp circuit in Fig. 7.131. Assume that $v(0) = -2$ V, $R = 10$ kΩ, and $C = 10\ \mu$F.

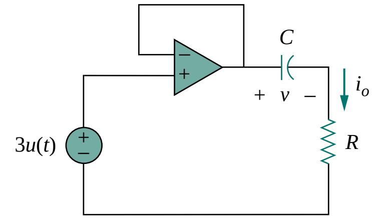


Figure 7.131 For Prob. 7.63.

7.64 For the op amp circuit of Fig. 7.132, let $R_1 = 10$ kΩ, $R_f = 20$ kΩ, $C = 20\ \mu$F, and $v(0) = 1$ V. Find v_o.

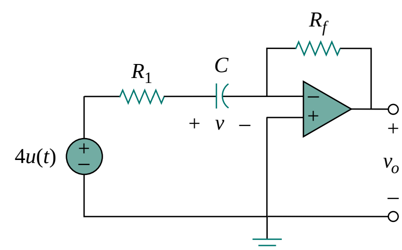


Figure 7.132 For Prob. 7.64.

7.65 Determine $v_o(t)$ for $t > 0$ in the circuit of Fig. 7.133. Let $i_s = 10u(t)\ \mu$A and assume that the capacitor is initially uncharged.

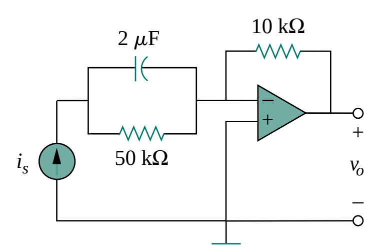


Figure 7.133 For Prob. 7.65.

7.66 In the circuit of Fig. 7.134, find v_o and i_o, given that $v_s = 4u(t)$ V and $v(0) = 1$ V.

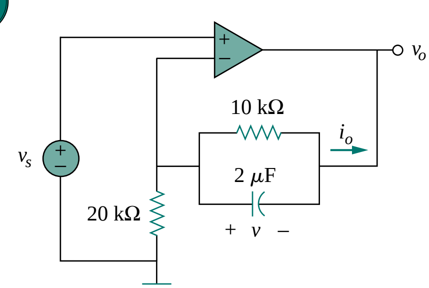


Figure 7.134 For Prob. 7.66.

Section 7.8 Transient Analysis with *PSpice*

7.67 Repeat Prob. 7.40 using *PSpice*.

7.68 The switch in Fig. 7.135 opens at $t = 0$. Use *PSpice* to determine $v(t)$ for $t > 0$.

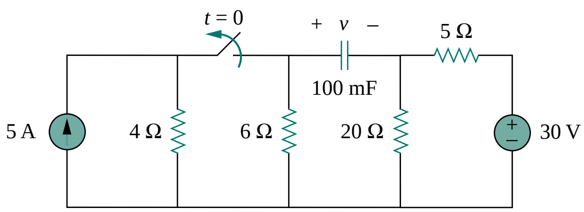


Figure 7.135 For Prob. 7.68.

7.69 The switch in Fig. 7.136 moves from position a to b at $t = 0$. Use *PSpice* to find $i(t)$ for $t > 0$.

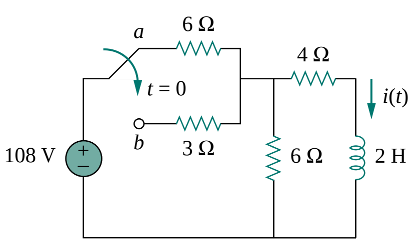


Figure 7.136 For Prob. 7.69.

7.70 Repeat Prob. 7.56 using *PSpice*.

Section 7.9 Applications

7.71 In designing a signal-switching circuit, it was found that a 100-μF capacitor was needed for a time constant of 3 ms. What value resistor is necessary for the circuit?

7.72 A simple relaxation oscillator circuit is shown in Fig. 7.137. The neon lamp fires when its voltage reaches 75 V and turns off when its voltage drops to 30 V. Its resistance is 120 Ω when on and infinitely high when off.

(a) For how long is the lamp on each time the capacitor discharges?

(b) What is the time interval between light flashes?

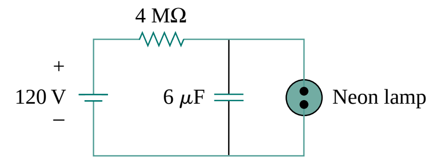


Figure 7.137 For Prob. 7.72.

7.73 Figure 7.138 shows a circuit for setting the length of time voltage is applied to the electrodes of a welding machine. The time is taken as how long it takes the capacitor to charge from 0 to 8 V. What is the time range covered by the variable resistor?

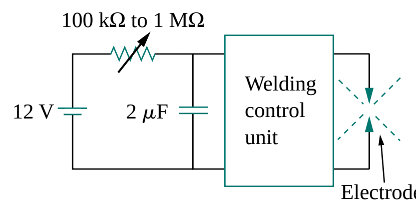


Figure 7.138 For Prob. 7.73.

7.74 A 120-V dc generator energizes a motor whose coil has an inductance of 50 H and a resistance of 100 Ω. A field discharge resistor of 400 Ω is connected in parallel with the motor to avoid damage to the motor, as shown in Fig. 7.139. The system is at steady state. Find the current through the discharge resistor 100 ms after the breaker is tripped.

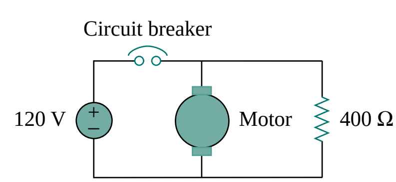


Figure 7.139 For Prob. 7.74.

COMPREHENSIVE PROBLEMS

7.75 The circuit in Fig. 7.140(a) can be designed as an approximate differentiator or an integrator, depending on whether the output is taken across the resistor or the capacitor, and also on the time constant $\tau = RC$ of the circuit and the width T of the input pulse in Fig. 7.140(b). The circuit is a differentiator if $\tau \ll T$, say $\tau < 0.1T$, or an integrator if $\tau \gg T$, say $\tau > 10T$.

(a) What is the minimum pulse width that will allow a differentiator output to appear across the capacitor?

(b) If the output is to be an integrated form of the input, what is the maximum value the pulse width can assume?

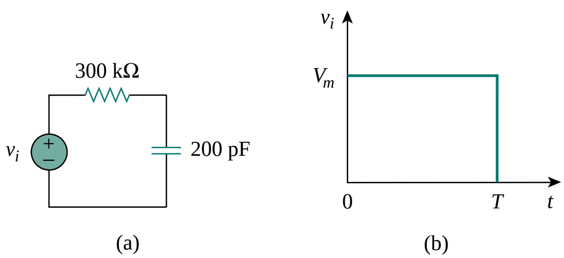


Figure 7.140 For Prob. 7.75.

7.76 An RL circuit may be used as a differentiator if the output is taken across the inductor and $\tau \ll T$ (say $\tau < 0.1T$), where T is the width of the input pulse. If R is fixed at 200 kΩ, determine the maximum value of L required to differentiate a pulse with $T = 10\ \mu$s.

7.77 An attenuator probe employed with oscilloscopes was designed to reduce the magnitude of the input voltage v_i by a factor of 10. As shown in Fig. 7.141, the oscilloscope has internal resistance R_s and capacitance C_s, while the probe has an internal resistance R_p. If R_p is fixed at 6 MΩ, find R_s and C_s for the circuit to have a time constant of 15 μs.

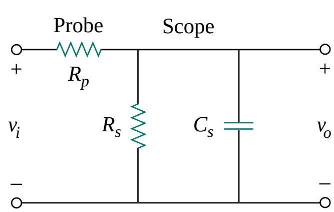


Figure 7.141 For Prob. 7.77.

7.78 The circuit in Fig. 7.142 is used by a biology student to study "frog kick." She noticed that the frog kicked a little when the switch was closed but kicked violently for 5 s when the switch was opened. Model the frog as a resistor and calculate its resistance. Assume that it takes 10 mA for the frog to kick violently.

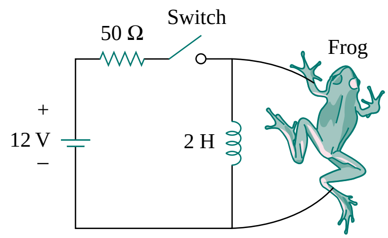


Figure 7.142 For Prob. 7.78.

7.79 To move a spot of a cathode-ray tube across the screen requires a linear increase in the voltage across the deflection plates, as shown in Fig. 7.143. Given that the capacitance of the plates is 4 nF, sketch the current flowing through the plates.

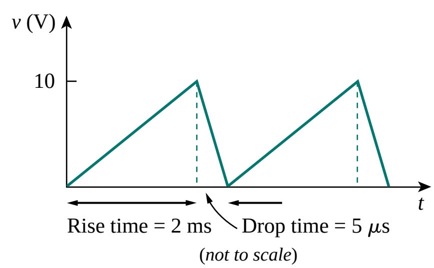


Figure 7.143 For Prob. 7.79.