C H A P T E R

4

Oscillating Systems

4.1 INTRODUCTION

We continue our investigation of oscillating systems by dividing our discussion into oscillations in electrical circuits and phase diagrams of nonlinear systems. To start, we extend the results derived for mechanical linear oscillating systems to electrical oscillating systems. There is a complete analogy between electrical and mechanical systems; hence we need not derive all the results. A portion of this chapter will be devoted to the study of nonlinear systems. We shall introduce several techniques for solving oscillating systems, as well.

Chapter 3 was completely devoted to the study of linear oscillating systems. This type of simple harmonic motion, although extensively used in physics and engineering problems, is truly applicable to only a limited number of cases: Nature does not allow such simplicity. Oscillating motion, in general, is nonlinear. A departure from linear motion occurs whenever the restoring force does not have a linear dependence on displacement; furthermore, the damping force may not have a linear dependence on velocity. Thus an equation representing the free motion of an oscillating system may be written as

$$m\ddot{x} + G(\dot{x}) + F(x) = 0 \tag{4.1}$$

where F is nonlinear function of x and G is a nonlinear function of $\dot{x}$.

For arbitrary functions F and G, the general solutions of Eq. (4.1) are not known. Equation (4.1) can be solved only for some particular cases. In general, we must use approximate methods to get some idea of the nature of the oscillations. Each particular situation must be treated as a special case and solved individually. We should keep in mind that the nonlinear nature of oscillating systems is, in general, due to the large amplitudes of oscillating systems.

To get approximate solutions, several techniques have been developed. It is not possible to treat them at any length in this chapter. We shall introduce the principle of superposition and the techniques of Fourier analysis, which are extremely helpful in solving nonlinear systems. We shall employ the method of Green's function whenever the system is acted on by a large force for a short interval of time. We shall extend our discussion to symmetrical and nonsymmetrical nonlinear systems and use the techniques of series expansion and successive approximations in solving such problems. We shall conclude the chapter with a qualitative discussion of these nonlinear systems employing the method of phase diagrams.

4.2 HARMONIC OSCILLATIONS IN ELECTRICAL CIRCUITS

There is a complete analogy between the free, damped, and forced oscillations of a single particle, which we discussed in Chapter 3, and several electrical circuits, which we discuss now. Furthermore, this analogy extends to many physical situations in nature, including atomic, molecular, and nuclear physics. Some time ago, electrical circuits were constructed by analogy with mechanical systems, but recently the situation has been reversed. Electrical circuit designs are now so advanced that mechanical engineers use them extensively in investigating mechanical vibrational problems.

Figure 4.1(i) shows the three mechanical systems that we have been discussing and Fig. 4.1(ii) shows the corresponding electrical oscillating systems. Table 4.1 shows the corresponding mechanical and electrical quantities in the oscillating systems. We shall discuss this analogy presently. To start, let us consider Fig. 4.1(a)(i), which shows a simple oscillator consisting of mass m tied to a spring of spring constant k. The mass moves on a frictionless surface and hence behaves like a free oscillator. Its motion is represented by the equation

$$m\ddot{x} + kx = 0 \qquad \text{or} \qquad \ddot{x} + \frac{k}{m}x = 0 \tag{4.2}$$

which has a solution

$$x = x_0 \cos \omega_0 t \tag{4.3a}$$

where ω_0 is the free natural frequency given by

$$\omega_0 = \sqrt{\frac{k}{m}} \tag{4.3b}$$

The electrical analogy to this, shown in Fig. 4.1(a)(ii), consists of a capacitor C and an inductor L. Let $Q = Q(t)$ be the charge on the capacitor at any time t and $I = I(t)$ be the current through the inductor. The relations between I and Q are

$$I = \frac{dQ}{dt} = \dot{Q} \tag{4.4a}$$

$$Q = \int I\, dt \tag{4.4b}$$

The voltage drop across the inductor is

$$V_L = L\frac{dI}{dt} \tag{4.5}$$

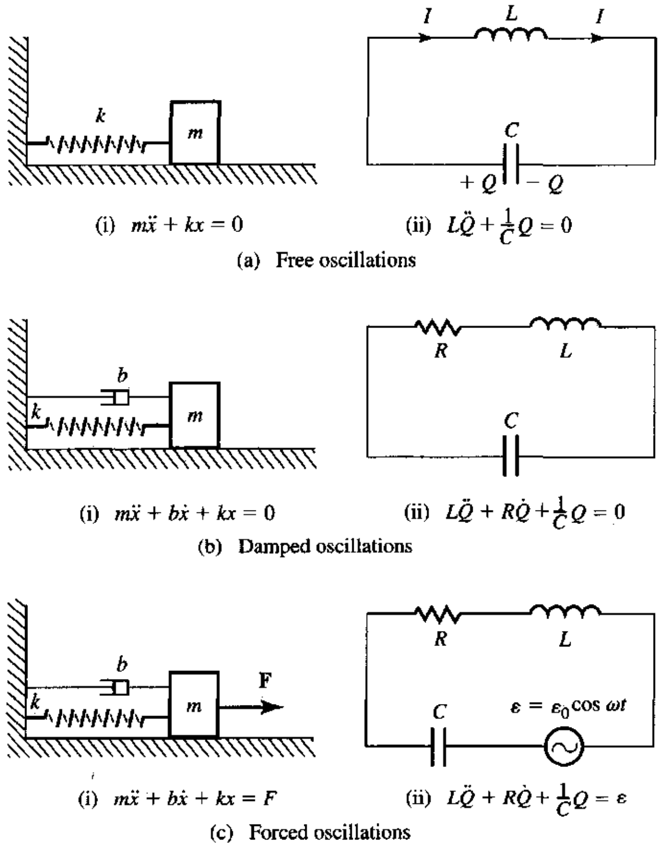


Figure 4.1 Analogy between (i) a mechanical system and (ii) an electrical system: (a) free oscillations, (b) damped oscillations, and (c) forced oscillations.

Table 4.1 Analogy between Mechanical and Electrical Quantities

Mechanical	Electrical	
Displacement	$x \leftrightarrow Q$	Charge
Velocity	$\dot{x} \leftrightarrow \dot{Q} = I$	Current
Mass	$m \leftrightarrow L$	Inductance
Compliance	$1/k \leftrightarrow C$	Capacitance
Damping constant	$b \leftrightarrow R$	Resistance
Applied force	$F \leftrightarrow \varepsilon$	Applied emf

and across the capacitor it is

$$V_C = \frac{Q}{C} = \frac{1}{C}\int I\,dt \tag{4.6}$$

Thus the sum of the voltages encountered in going around the whole circuit must be zero; that is, by applying Kirchhoff's rule, we get

$$V_L + V_C = 0$$

or

$$L\frac{dI}{dt} + \frac{1}{C}\int I\,dt = 0 \tag{4.7}$$

Using Eqs. (4.4), we get

$$L\ddot{Q} + \frac{1}{C}Q = 0 \quad \text{or} \quad \ddot{Q} + \frac{1}{LC}Q = 0 \tag{4.8}$$

This equation is identical to the mechanical system given by Eq. (4.2) provided

$$Q \leftrightarrow x, \quad \ddot{Q} \leftrightarrow \ddot{x}, \quad L \leftrightarrow m, \quad \text{and} \quad C \leftrightarrow 1/k$$

Assuming $Q = Q_0$ at $t = 0$, the solution is

$$Q = Q_0 \cos \omega_0 t \tag{4.9}$$

with the free natural frequency

$$\omega_0 = \sqrt{\frac{1}{LC}} \tag{4.10}$$

If we differentiate Eq. (4.9), we may write

$$\frac{dQ}{dt} = I = -\omega_0 Q_0 \sin \omega_0 t = -I_0 \sin \omega_0 t \tag{4.11}$$

Let us now consider the damped oscillator shown in Fig. 4.1(b)(i). Note that we have added a dashpot filled with viscous fluid to indicate damping with damping constant b. The differential equation representing the motion of a damped oscillator is

$$m\ddot{x} + b\dot{x} + kx = 0 \tag{4.12}$$

while Fig. 4.1(b)(ii) shows the equivalent electrical circuit, where we have added a resistor R, which is equivalent to the damping or frictional force in mechanical systems. The electrical equation analogous to a mechanical system is

$$L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = 0 \tag{4.13}$$

Figure 4.1(c)(i) shows a mechanical driven oscillator represented by the following equation:

$$m\ddot{x} + b\dot{x} + kx = F = F_0 \cos \omega t \tag{4.14}$$

Figure 4.1(c)(ii) represents an analogous electrical driven oscillator with an emf source given by $\varepsilon = \varepsilon_0 \cos \omega t$. The corresponding equation is

$$L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = \varepsilon = \varepsilon_0 \cos \omega t \tag{4.15}$$

To extend our analogy still further we give another example, shown in Fig. 4.2(a) and (b). Suppose we apply a force F to a system consisting of two springs in line [Fig. 4.2(a)(i)]. The net displacement x is given by the sum of the displacement x_1 and x_2 caused by the two springs. Thus

$$x = x_1 + x_2 = \frac{F}{k_1} + \frac{F}{k_2} = F\left(\frac{1}{k_1} + \frac{1}{k_2}\right) = \frac{F}{k}$$

That is, when the *springs* are *in series*, the equivalent k is given by

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} \tag{4.16}$$

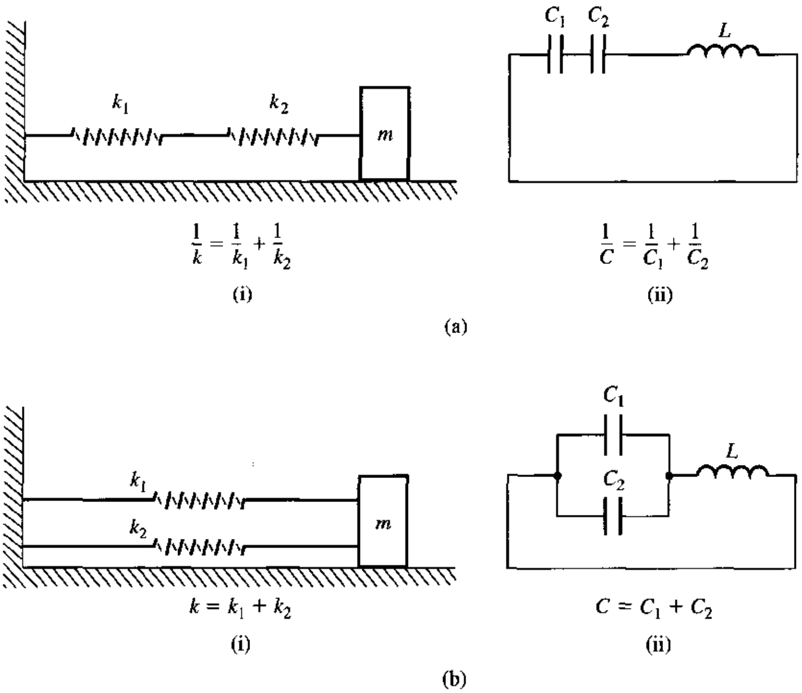


Figure 4.2 Analogy between mechanical and electrical systems: (a) series systems, and (b) parallel systems.

When the *springs* are *in parallel* [Fig. 4.2(b)(i)],

$$k = k_1 + k_2 \tag{4.17}$$

When the *capacitors* are *in series* [Fig. 4.2(a)(ii)], the equivalent capacitance C is

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} \tag{4.18}$$

and when the *capacitors* are *in parallel* [Fig. 4.2(b)(ii)], the equivalent capacitance C is given by

$$C = C_1 + C_2 \tag{4.19}$$

Energy Considerations

For a free oscillator the total energy—the sum of the kinetic and potential energy—is always constant as long as there is no damping and is given by

$$\tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2 = \text{constant} \tag{4.20}$$

Either by analogy with this equation or from Table 4.1, and using Eqs. (4.9) and (4.11), we get

$$\frac{1}{2}LI^2 + \frac{1}{2}\frac{Q^2}{C} = \left(\frac{Q_0^2}{2C}\right) = \text{constant} \tag{4.21}$$

$\frac{1}{2}LI^2$ is the energy stored in an inductor and is equivalent to the mechanical kinetic energy $\frac{1}{2}mv^2$; $\frac{1}{2}Q^2/C$ is the energy stored in the capacitor and is equivalent to the mechanical potential energy $\frac{1}{2}kx^2$.

Let us extend this analogy to a damped oscillator. Starting with Eq. (4.13),

$$L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = 0$$

and multiplying both sides by $\dot{Q}$ yields

$$L\ddot{Q}\dot{Q} + R\dot{Q}^2 + \frac{Q}{C}\dot{Q} = 0$$

or

$$\frac{d}{dt}\left(\frac{1}{2}L\dot{Q}^2\right) + \frac{d}{dt}\left(\frac{1}{2}\frac{Q^2}{C}\right) = -R\dot{Q}^2 \tag{4.22}$$

Since $\dot{Q} = I$, we may write

$$\frac{d}{dt}\left(\frac{1}{2}LI^2 + \frac{1}{2}\frac{Q^2}{C}\right) = -RI^2 \tag{4.23}$$

This equation states that the rate at which the energy is being stored in the inductor and the capacitor is equal to the energy dissipated in the resistor, as it should be. For mechanical systems, Eq. (4.22), after proper substitution, takes the following form:

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\right) = -b\dot{x}^2 \tag{4.24}$$

The preceding analogy can be further extended to a driven oscillator, getting its solution and then writing an expression for average power dissipation, resonance, and quality factor.

4.3 PRINCIPLE OF SUPERPOSITION AND FOURIER SERIES

The principle of superposition is used throughout physics. Most students encounter this principle in the following simple form while studying wave motion and optics in general physics.

> ***Principle of Superposition.*** When two or more waves travel simultaneously through a portion of a medium, each wave acts independently as if the other were not present. The resultant displacement at any point is the vector sum of the displacements of the individual waves.

We now extend this principle to the case of harmonic oscillators and to linear operators in general. If $x_1(t), x_2(2), \ldots$ are the solutions when the forces acting are $F_1(t), F_2(t), \ldots$, respectively, then $x(t) = x_1(t) + x_2(t) + \cdots$ is a solution when the force acting on the system is $F(t) = F_1(t) + F_2(t) + \cdots$. We may further generalize as follows. The second-order linear differential equation describing a forced harmonic oscillator given by

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = F(t) \tag{4.25}$$

may be written in general form as

$$\left(\frac{d^2}{dt^2} + a\frac{d}{dt} + b\right)x(t) = F(t) \tag{4.26}$$

We define a *linear operator, L*, as the quantity in the parentheses on the left; that is,

$$L \equiv \left(\frac{d^2}{dt^2} + a\frac{d}{dt} + b\right) \tag{4.27}$$

Thus Eq. (4.25) or (4.26) may be written as

$$Lx(t) = F(t) \tag{4.28}$$

According to the *superposition principle:*

> *If a set of functions $x_n(t)$, $n = 1, 2, 3, \ldots$, comprises solutions of a linear differential equation*
>
> $$Lx_n(t) = F_n(t) \tag{4.29}$$
>
> *then the function $x(t)$, which is a linear combination of $x_n(t)$, that is,*
>
> $$x(t) = \sum_n C_n x_n(t) \tag{4.30}$$
>
> *where C_n are constants, satisfies the differential equation*
>
> $$Lx(t) = F(t) \tag{4.31}$$

where

$$F(t) = \sum_n C_n F_n(t) \tag{4.32}$$

We can prove this statement by substituting Eq. (4.30) into Eq. (4.31); that is,

$$Lx(t) = L\left[\sum_n C_n x_n(t)\right] = \sum_n C_n Lx_n(t) = \sum_n C_n F_n(t) = F(t)$$

as it should be.

Let us apply these results to the general case of the driven harmonic oscillator we have discussed in detail. Suppose the individual driving forces $F_n(t)$ have a harmonic dependence of the form $\cos(\omega_n t - \theta_n)$, so that

$$F(t) = \sum_n C_n \cos(\omega_n t - \theta_n) \tag{4.33}$$

When the force was of the form $F_0 \cos(\omega t + \theta_0)$, the steady-state solution was given by Eqs. (3.99) and (3.100). Thus for $F(t)$ given by Eq. (4.33), the steady-state solution is

$$x(t) = \sum_n \frac{C_n}{m} \frac{1}{\sqrt{(\omega_0^2 - \omega_n^2)^2 + 4\gamma^2 \omega_n^2}} \cos(\omega_n t - \theta_n - \phi_n) \tag{4.34}$$

where

$$\phi_n = \tan^{-1}\left(\frac{2\gamma\omega_n}{\omega_0^2 - \omega_n^2}\right) \tag{4.35}$$

The general solution is the sum of the transient and the steady state and is given by

$$x(t) = Ae^{-\gamma t} \cos(\omega_1 t + \theta') + \sum_n \frac{C_n}{m} \frac{\cos(\omega_n t - \theta_n - \phi_n)}{\sqrt{(\omega_0^2 - \omega_n^2)^2 + 4\gamma^2\omega_n^2}} \tag{4.36}$$

where the constants A and θ' are to be determined from initial conditions as usual. Similar results can be obtained if $F(t)$ has a dependence of the form $\sin(\omega_n t - \theta_n)$.

With the help of the Fourier theorem we can extend the preceding type of consideration to the case in which the driving force is periodic (but not harmonic) and is a continuous or piecewise continuous function. $F(t)$ is a periodic function if it satisfies the condition

$$F(t + T) = F(t) \tag{4.37}$$

where $T = 2\pi/\omega$ is the period of the applied force. According to the ***Fourier theorem,*** *any arbitrary periodic function, which is continuous or piecewise continuous, having only a finite number of discontinuities over a time period, can be expressed as a sum of harmonic terms.* Thus any function $F(t)$ that is defined within a time interval $-T/2 < t < T/2$ [or a function $F(x)$ within a time interval $-\pi < x < \pi$] can be expressed as a series of sine and cosine terms as

$$F(t) = \frac{A_0}{2} + A_1 \cos \omega t + A_2 \cos 2\omega t + \cdots + A_n \cos n\omega t + \cdots$$
$$+ B_1 \sin \omega t + B_2 \sin 2\omega t + \cdots + B_n \sin n\omega t + \cdots \tag{4.38a}$$

where A_n and B_n are constants and $n = 1, 2, 3, \ldots, n, \ldots, \infty$. Equation (4.38a) may also be written as

$$F(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty}(A_n \cos n\omega t + B_n \sin n\omega t) \tag{4.38b}$$

The terms in the sum (on the right) form a *Fourier series.* The constants A_n and B_n can be evaluated by integration. For example, to evaluate A_n we multiply both sides of Eq. (4.38b) by $\cos m\omega t$ (m being an integer) and integrate between the limits $-T/2$ and $T/2$:

$$\int_{-T/2}^{+T/2} F(t) \cos m\omega t \, dt = \frac{A_0}{2}\int_{-T/2}^{+T/2} \cos m\omega t \, dt$$

$$+ \sum_{n=1}^{\infty}\left[A_n \int \cos n\omega t \cos m\omega t \, dt + B_n \int \sin n\omega t \cos m\omega t \, dt\right]$$

Since m and n are integers, for all values of m and n, we obtain

$$\int_{-T/2}^{+T/2} \cos n\omega t \cos m\omega t \, dt = 0, \qquad \text{if } m \neq n \text{ and } = T/2 \text{ if } m = n$$

$$\int_{-T/2}^{+T/2} \sin n\omega t \cos m\omega t \, dt = 0, \qquad \text{for all values of } m \text{ and } n$$

$$\int_{-T/2}^{+T/2} \cos m\omega t \, dt = 0, \qquad \text{if } m \neq 0 \text{ and } = T \text{ if } m = 0$$

Thus the values of A_0 and A_n are

$$A_0 = \frac{2}{T}\int_{-T/2}^{+T/2} F(t)\, dt \tag{4.39a}$$

$$A_n = \frac{2}{T}\int_{-T/2}^{+T/2} F(t) \cos n\omega t \, dt \qquad \text{if } n \text{ is an integer} \tag{4.39b}$$

That is, $n = 1, 2, 3, \ldots$. Similarly multiplying Eq. (4.38b) by $\sin m\omega t$ and integrating, we obtain

$$B_n = \frac{2}{T}\int_{-T/2}^{+T/2} F(t) \sin n\omega t \, dt \qquad \text{if } n \text{ is an integer} \tag{4.39c}$$

That is, $n = 1, 2, 3, \ldots$. If necessary we can replace the integration limits $-T/2$ $(= -\pi/\omega)$ to $T/2$ $(= +\pi/\omega)$ by 0 to T $(=2\pi/\omega)$.

First one has to determine the appropriate number of terms that must be used in the Fourier series to approximate the arbitrary driving force. This is illustrated in two cases: (1) a rectangular function, and (2) a sawtooth function, as we shall discuss shortly. Once we know the series, each term used in the applied force has a corresponding solution. By adding all these solutions we obtain the general solution of a damped harmonic oscillator driven by an arbitrary force. In actual practice, obtaining solutions by this method is quite tedious, but in some situations it is helpful.

Example 4.1

Consider the function F(θ) given by the equations and graphed below. Find a Fourier series expansion of the function F(θ), where θ (= ω t) is a function of time t and angular velocity ω.

Solution

Redefine function F(θ) in terms of function f(θ). Graph f(θ) for the different values of θ given below.

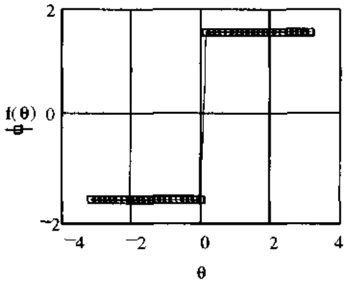
$F(\theta) = -\frac{\pi}{2}$ between $-\pi \le \theta \le 0$

$F(\theta) = \frac{\pi}{2}$ between $0 \le \theta \le \pi$

$\theta := -\pi, -19 \cdot \frac{\pi}{20} .. \pi \qquad F(\theta) := \frac{\pi}{2}$

$f(\theta) := \mathrm{if}(\theta > 0, F(\theta), -F(\theta))$

In order to find the Fourier series expansion that will result in function f(θ), we evaluate the coefficients A0, A, and B by using Eq. (4.39). Let T be the time period. As t changes from -T/2 to T/2, θ changes from -π to π.



$n := 1, 2 .. 10$

$$A0 := -\frac{1}{2} \cdot \int_{-\pi}^{0} 1 \, d\theta + \frac{1}{2} \cdot \int_{0}^{\pi} 1 \, d\theta$$

$$A_n := -\frac{1}{2} \cdot \int_{-\pi}^{0} \cos(n \cdot \theta) \, d\theta + \frac{1}{2} \cdot \int_{0}^{\pi} \cos(n \cdot \theta) \, d\theta \qquad B_n := -\frac{1}{2} \cdot \int_{-\pi}^{0} \sin(n \cdot \theta) \, d\theta + \frac{1}{2} \cdot \int_{0}^{\pi} \sin(n \cdot \theta) \, d\theta$$

$A0 = 0$

A_n	B_n
0	2
0	0
0	0.667
0	0
0	0.4
0	0
$-1.147 \cdot 10^{-15}$	0.286
0	0
0	0.222
0	0

The values of the first two are

$A0 = 0 \qquad A_n = 0$

For B_n even terms are 0, hence the expansion series consists only of odd sine terms.

$$F(\theta) = \frac{2}{1} \cdot \sin(\theta) + \frac{2}{3} \cdot \sin(3 \cdot \theta) + \frac{2}{5} \cdot \sin(5 \cdot \theta) + \ldots\ldots$$

As an example, H11, H12, and H13 represent series using n = 1, 2, and 3 terms, respectively.

$$H11_n = \frac{A0}{2} + \frac{2}{1} \cdot \theta_n \qquad H12_n = \frac{A0}{2} + \frac{2}{1} \cdot \sin(\theta_n) + \frac{2}{3} \cdot \sin(3 \cdot \theta_n)$$

$$H13_n = \frac{A0}{2} + \frac{2}{1} \cdot \sin(\theta_n) + \frac{2}{3} \cdot \sin(3 \cdot \theta_n) + \frac{2}{5} \cdot \sin(5 \cdot \theta_n)$$

Now we change the range number n to the desired value. We plot the functions H (replacing F) using different numbers of terms. H1 means 1 term, H3 means three terms, H5 means 5 terms, and so on. It is clear from the graph below that the H function approaches the F function as the number of terms increases. For example, using n = 15 terms, H almost coincides with the f(θ) plot. Using more terms will make the graph still closer to the rectangular graph.

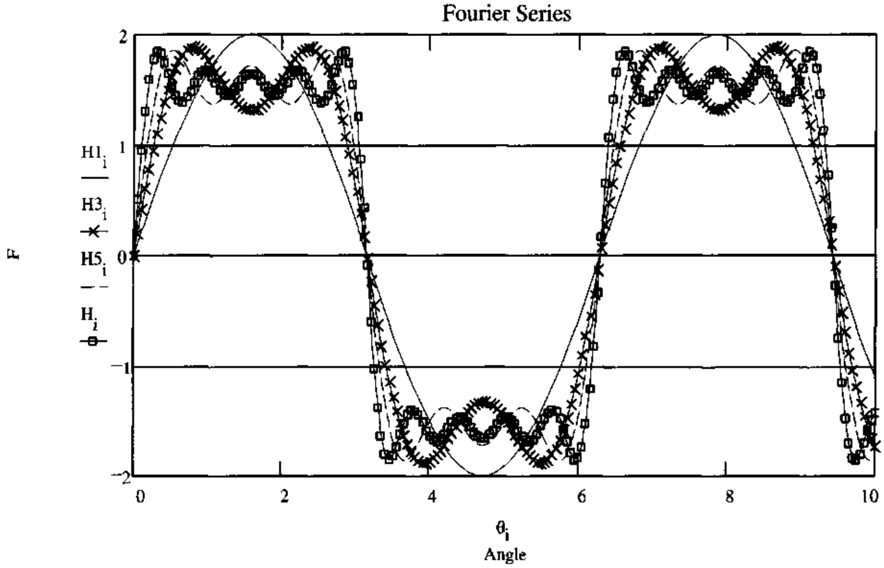
$$n := 0..15 \qquad I := 200 \qquad i := 0..I \qquad \theta_i := \frac{i}{20}$$

$$H1_i := \frac{A0}{2} + \left[\sum_{n=1}^{1} \left(A_n \cdot \cos(n \cdot \theta_i) + B_n \cdot \sin(n \cdot \theta_i)\right)\right]$$

$$H3_i := \frac{A0}{2} + \left[\sum_{n=1}^{3} \left(A_n \cdot \cos(n \cdot \theta_i) + B_n \cdot \sin(n \cdot \theta_i)\right)\right]$$

$$H5_i := \frac{A0}{2} + \left[\sum_{n=1}^{5} \left(A_n \cdot \cos(n \cdot \theta_i) + B_n \cdot \sin(n \cdot \theta_i)\right)\right]$$

$$H_i := \frac{A0}{2} + \left[\sum_{n=1}^{10} \left(A_n \cdot \cos(n \cdot \theta_i) + B_n \cdot \sin(n \cdot \theta_i)\right)\right]$$



(a) Write H5 in terms of an expansion series.

(b) How will the graph look if A0 is not zero but is constant? Draw the graphs.

(c) Graph for 30 terms and 50 terms. What results do you expect?

(d) How will the plot of H versus θ differ from H versus t?

EXERCISE 4.1 Consider the function shown in Fig. Exer. 4.1 in the interval $-\pi < \theta < \pi$. Find a Fourier series expansion of this function.

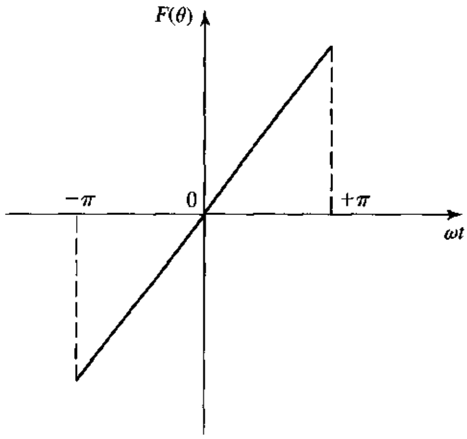


Figure Exer. 4.1

4.4 HARMONIC MOTION AND GREEN'S FUNCTION

Impulsive Force

When a very large force $F(t)$ acts on a system for a very short interval of time Δt, an *impulse* is said to be imparted to the system. It is the same thing when a force $F(t)$ applied to a system has a small value for a short interval of Δt, while almost negligible before and after this interval. By applying the impulse–momentum theorem and the superposition principle to the oscillating system, we can arrive at many interesting and useful results. According to the *impulse–momentum theorem,*

$$p_f - p_i = F\,\Delta t = \int F\,dt \tag{4.40a}$$

or

$$\Delta p = F\,\Delta t \tag{4.40b}$$

and

$$\Delta v = \frac{F}{m}\Delta t \tag{4.40c}$$

We consider the application of this to several situations.

Case (i) An Oscillator Initially at Rest: To start, let us assume that we are dealing with an undamped oscillator at rest; that is, $x = 0$ and $\dot{x} = 0$. At time $t = t_0$ an impulse is given to the oscillator so that its velocity right after the impulse is v_0. Thus from Eq. (4.40b),

$$\Delta p = mv_0 = F\,\Delta t \tag{4.41a}$$

or

$$v_0 = \frac{F}{m}\Delta t \tag{4.41b}$$

Since at $t = t_0$, $x = 0$ the displacement x of an undamped oscillator is

$$x = A \sin[\omega_0(t - t_0)] \tag{4.42a}$$

where A is to be determined from initial conditions. (Note that we have neglected any short displacement that may result during a short interval Δt when the force is applied.) Differentiating Eq. (4.42a) and substituting $\dot{x} = v_0$ when $t = t_0$, we get

$$\dot{x} = \omega_0 A \cos[\omega_0(t - t_0)] \tag{4.42b}$$

and

$$v_0 = \omega_0 A$$

Therefore,

$$A = \frac{v_0}{\omega_0} = \frac{F\,\Delta t}{m\omega_0} = \frac{\Delta p}{m\omega_0} \tag{4.43}$$

Thus the general solution is

$$x(t) = \begin{cases} 0, & \text{for } t \leqslant t_0 \\ \dfrac{F\,\Delta t}{m\omega_0} \sin[\omega_0(t - t_0)], & \text{for } t \geqslant t_0 \end{cases} \tag{4.44}$$

This procedure can be easily extended to the case of damped harmonic oscillators at rest, for which

$$x = Ae^{-\gamma(t-t_0)} \sin[\omega_1(t - t_0)] \tag{4.45}$$

and the final solution after the impulse has been applied is

$$x(t) = \begin{cases} 0, & \text{for } t < t_0 \\ \dfrac{F\,\Delta t}{m\omega_1} e^{-\gamma(t-t_0)} \sin[\omega_1(t - t_0)], & \text{for } t \geqslant t_0 \end{cases} \tag{4.46}$$

Case (ii) An Oscillator Initially Not at Rest: Once again let us start with an undamped oscillator, but this time initially it has a displacement x_0 and velocity v_0. Thus, starting with

$$x = B \cos \omega_0 t + C \sin \omega_0 t \tag{4.47}$$

and applying the initial conditions at $t = t_0$, $x = x_0$, and $\dot{x} = v_0$, we get

$$x = x_0 \cos \omega_0(t - t_0) + \frac{v_0}{\omega_0} \sin \omega_0(t - t_0) \tag{4.48}$$

Let us now apply a force $F(t)$ at $t = t_0$ for short interval Δt. According to the impulse–momentum theorem,

$$\Delta p = F\,\Delta t \qquad \text{or} \qquad \Delta v = F\frac{\Delta t}{m} \tag{4.49}$$

where Δv is a small additional velocity given to the system, which already has some velocity v_0 at time t_0. The additional displacement resulting from the impulse may be calculated as if Δv

were the initial velocity; that is, by replacing v_0 in Eq. (4.48) by Δv given by Eq. (4.49), we get the additional displacement x_1 to be

$$x_1 = \frac{F\,\Delta t}{m\omega_0} \sin \omega_0(t - t_0) \tag{4.50}$$

Thus the total displacement is the sum of x and x_1 given by Eqs. (4.48) and (4.50):

$$x(t) = x_0 \cos \omega_0(t - t_0) + \frac{v_0}{\omega_0} \sin \omega_0(t - t_0) + \frac{F\,\Delta t}{m\omega_0} \sin \omega_0(t - t_0) \tag{4.51}$$

We can extend this treatment to the case of a damped harmonic oscillator that at $t = t_0$ has $x = x_0$ and $\dot{x} = v_0$, while an impulse is given at $t = t_0$. (See Problem 4.23.) Before the impulse the motion is described by

$$x = e^{-\gamma t}\,[B \cos \omega_1 t + C \sin \omega_1 t] \tag{4.52}$$

After the impulse the motion is described by

$$xt = e^{-\gamma(t-t_0)}\left(x_0 \cos \omega_1(t - t_0) + \frac{v_0}{\omega_1} \sin \omega_1(t - t_0)\right) + \frac{F\,\Delta t}{m\omega_1} e^{-\gamma(t-t_0)} \sin[\omega_1(t - t_0)] \tag{4.53}$$

Continuous Arbitrary Force and Green's Function

As long as we are considering a linear oscillator, we can extend the application of the impulse–momentum theorem to the case of an arbitrary force function. The solution is based on the method developed by George Green. According to *Green's method, an arbitrary force function $F(t)$ can be thought of as a series of impulses, each acting for a short interval of time Δt and delivering an impulse $F(t)\,\Delta t$,* as shown in Fig. 4.3. Thus

$$F(t) = \sum_{n=-\infty}^{n=+\infty} F_n(t) \tag{4.54}$$

where

$$F_n(t) \left| \begin{array}{ll} = F(t_n), & \text{if } t_n < t < t_{n+1} \\ = 0, & \text{if } t < 0 \text{ or } t > t_{n+1} \end{array} \right. \tag{4.55}$$

and $t = n\,\Delta t$. It is clear from Fig. 4.3 that if $\Delta t \to 0$, the sum of the series of impulses $\Sigma\, F_n(t)$ approaches $F(t)$. If the system is linear, we can always apply the principle of superposition. This allows us to write the inhomogeneous part of the differential equation as the sum of the individual impulses. That is, for

$$m\ddot{x} + b\dot{x} + kx = \sum F_n(t) \tag{4.56}$$

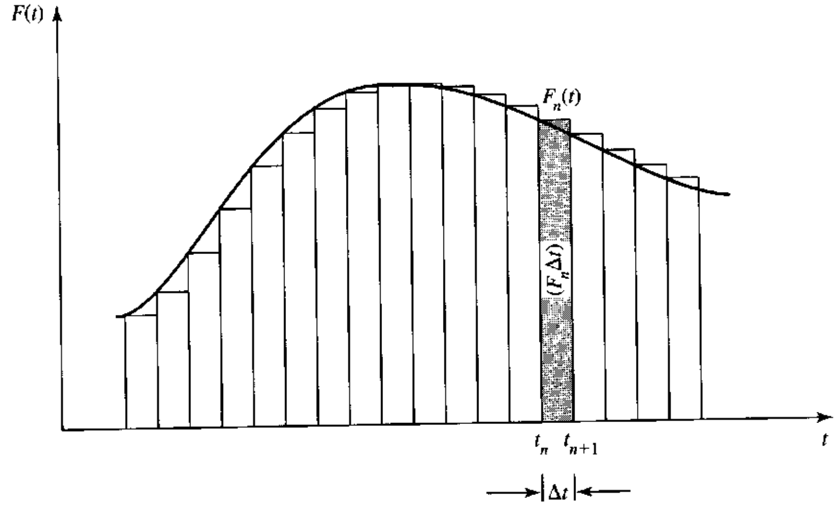


Figure 4.3 Arbitrary force function as a sum of a series of impulses.

the general steady-state solution is the sum of the individual solutions resulting from each $F_n(t)$. The individual solutions are of the type given by Eq. (4.46) for a single impulse. Thus the steady-state solution of Eq. (4.56) is

$$x(t) = \sum_{n=-\infty}^{N} \frac{F(t_n)\,\Delta t}{m\omega_1} e^{-\gamma(t-t_n)} \sin[\omega_1(t - t_n)] \tag{4.57}$$

which includes all solutions up to and including the Nth impulse. We can replace the summation by integration when $\Delta t \to 0$ and $t_n = t'$. That is,

$$x(t) = \int_{-\infty}^{t} \frac{F(t')}{m\omega_1} e^{-\gamma(t-t')} \sin[\omega_1(t - t')]\,dt' \tag{4.58}$$

We define *Green's function* $G(t, t')$ as

$$\begin{aligned} G(t, t') &= \frac{e^{-\gamma(t-t')}}{m\omega_1} \sin[\omega_1(t - t')], \qquad &&\text{for } t \geqslant t' \\ &= 0, &&\text{for } t < t' \end{aligned} \tag{4.59}$$

Thus, in terms of Green's function, we may write the steady-state solution [Eq. (4.58)] as

$$x(t) = \int_{-\infty}^{t} F(t')G(t, t')\,dt' \tag{4.60}$$

The main advantage of this method is that the solution is already adjusted for initial conditions; for example, in this case it is for a damped oscillator that is initially at rest at the equilibrium position. We must add the transient solution to the steady-state solution [Eq. (4.60)] to obtain a complete solution. For different initial conditions, solutions may be obtained by the same procedure. (See Problems 4.24 and 4.25.)

Example 4.2

A damped oscillator is acted on by the force function below. Using Green's function, graph the response function as well as the applied force.

$F(t) = 0$ if $t < 0$ $\qquad$ $F(t) = Fo \exp(-\gamma t) \sin(\omega t)$ if $t > 0$

Solution

Let us assume the values of the different variables are

$i := 0 .. 100 \qquad t_i := \frac{i}{5} \qquad F0 := 10 \qquad \gamma := .2$

$\omega := 2 \qquad \omega 1 := 1 \qquad M := .5$

The acting force F(t) may be written as (This means that if t < 0, F(t) is zero; otherwise graph the third term in ( ).)

$$F(t) := \text{if}\left(t<0, 0, F0 \cdot e^{-\gamma t} \cdot \sin(\omega \cdot t)\right)$$

Using Eq. (4.59), the corresponding Green's function may be written as

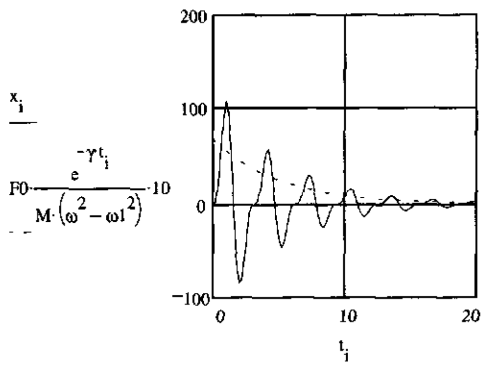
$$G(t,t') := \text{if}\left[t<0, 0, F0 \cdot \frac{e^{-\gamma(t-t')}}{M \cdot \omega 1} \cdot \sin(\omega \cdot t)\right]$$

The resulting displacement x is given by

$$x_i := \int_0^{t_i} F(t') \cdot G(t_i, t')\, dt'$$

The graphs of x and F versus t are as shown.

How do you explain the variation in x with time from the nature of the applied force?



EXERCISE 4.2 Repeat the above example for the case in which the sine term in the force function is replaced by a cosine term. What difference do you observe in the two graphs?

Example 4.3

Consider a damped harmonic oscillator that is suddenly acted on at $t = 0$ by a decaying force

$$F(t) = F_0\, e^{-kt}, \qquad t > 0$$

Find the general solution by using Green's function. (A typical example of such a force is the decay voltage on a capacitor.)

Solution

Since

$$F(t) = F(t') = F_0 e^{-kt'} \tag{i}$$

using Green's function, Eq. (4.59), the general solution, Eq. (4.60) is

$$x(t) = \int_0^t F(t')G(t, t')\, dt' = \frac{1}{m\omega_1}\int_0^t F_0\, e^{-kt'}\, e^{-\gamma(t-t')} \sin[\omega_1(t - t')]\, dt' \tag{ii}$$

If we substitute

$$y = \omega_1(t - t'), \quad dy = -\omega_1\, dt', \quad \text{or} \quad dt' = -\frac{dy}{\omega_1} \tag{iii}$$

then the limits change from $0 \to t$ to $\omega_1 t \to 0$. Therefore,

$$x(t) = \frac{F_0}{m\omega_1^2}\int_0^{\omega_1 t} e^{-kt}\, e^{(k-\gamma)y/\omega_1} \sin y\, dy \tag{iv}$$

In order to plot x versus t, we first solve this equation for x. Solving for xt and simplifying, the resulting equation is

$$xt = \frac{F0}{m\cdot\omega1^2}\cdot\int_0^{\omega1\cdot t} e^{-k\cdot t}\cdot e^{\frac{k-\gamma}{\omega1}\cdot\gamma}\cdot\sin(y)\, dy$$

$$x_t = -F0\cdot\frac{(\omega1\cdot\cos(\omega1\cdot t) - \sin(\omega1\cdot t)\cdot k + \sin(\omega1\cdot t)\cdot\gamma)}{\left[m\cdot\left[\omega1\cdot\left(k^2 - 2\cdot k\cdot\gamma + \gamma^2 + \omega1^2\right)\right]\right]}\cdot\exp(-t\cdot\gamma) + F0\cdot\frac{\exp(-k\cdot t)}{\left[m\cdot\left(k^2 - 2\cdot k\cdot\gamma + \gamma^2 + \omega1^2\right)\right]}$$

Using $\omega1$ and A given below, x can be written as

$$\omega1 = \sqrt{\omega0^2 - \gamma^2} \qquad A = \frac{\frac{F0}{m}}{(k-\gamma)^2 + \omega1^2} \qquad x_t = A\cdot\left[e^{-k\cdot t} - e^{-\gamma t}\cdot\left(\cos(\omega1\cdot t) - \frac{k-\gamma}{\omega1}\cdot\sin(\omega1\cdot t)\right)\right] \tag{v}$$

Below we graph x versus t for 30 values of t and for 3 values of the damping constant γ.

$N := 30 \qquad t := 0..N \qquad n := 1..3 \qquad k := .4 \qquad F0 := 100 \qquad m := .5$

$$\omega0 := \sqrt{\frac{k}{m}} \qquad b_n := \begin{array}{|c|}\hline .2 \\ \hline .4 \\ \hline 1 \\ \hline \end{array} \qquad \gamma_n := \frac{b_n}{2 \cdot m} \qquad \gamma_n = \begin{array}{|c|}\hline 0.2 \\ \hline 0.4 \\ \hline 1 \\ \hline \end{array} \qquad \omega1_n := \sqrt{\omega0^2 - (\gamma_n)^2}$$

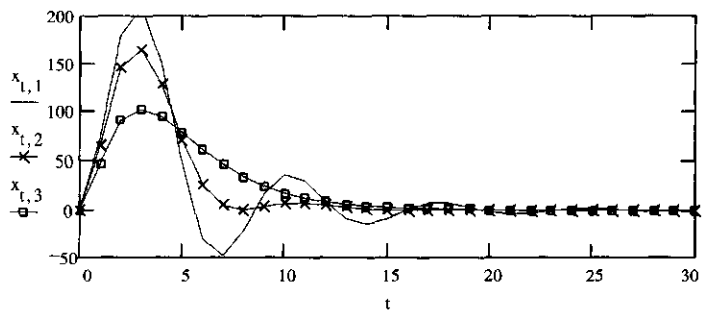
$\omega0 = 0.894$

$$A_n := \frac{\frac{F0}{m}}{(k - \gamma_n)^2 + (\omega1_n)^2} \qquad x_{t,n} := A_n \cdot \left[e^{-k \cdot t} - e^{-\gamma_n \cdot t} \cdot \left(\cos(\omega1_n \cdot t) - \frac{k - \gamma_n}{\omega1_n} \cdot \sin(\omega1_n \cdot t) \right) \right]$$

x1 for $k > \gamma$

x2 for $k = \gamma$

x3 for $k < \gamma$



(a) If $k \gg \gamma$ and both are small compared to ω_0, $[\omega_1 = (\omega_0^2 - \gamma^2)^{1/2}]$ the term e^{-kt} has an effect only for a short time in the beginning.

(b) When $k = \gamma$ and both are small compared to ω_0, Eq. (v) takes the form

$$x(t) = \frac{F_0}{m\omega_1^2} e^{-\gamma t}(1 - \cos \omega_1 t) \qquad \textbf{(vi)}$$

This means that the response function is still oscillatory, but with an exponentially decaying amplitude, as shown.

(c) If $k \ll \gamma$, the forcing function $F(t)$ given by Eq. (i) takes over the oscillatory motion; that is, the amplitude of the oscillations starts decaying exponentially after an initial increase over a short interval of time, as shown.

EXERCISE 4.3 Discuss the example if the decaying force at $t = 0$ is $F(t) = -kt^2$.

4.5 NONLINEAR OSCILLATING SYSTEMS

Before starting with this section, it will be worthwhile to review Section 3.2, where we explained the difference between linear and nonlinear systems. We have already seen that linear systems (systems in which the force is negatively proportional to the displacement) lead to harmonic oscillations (oscillations of one frequency). The condition imposed was that the motion must be limited to a small region near the equilibrium point.

Let us now consider systems in which the motion is not restricted and hence the restoring force is not proportional to the displacement. In general,

$$m\ddot{x} + F(x) = 0 \tag{4.61}$$

where $F(x)$ is the restoring force and is no longer linear. If damping is present, we shall have another free function $G(x)$, which may also be nonlinear. One outstanding characteristic of a nonlinear system is that, unlike linear systems, the time period of nonlinear systems, in general, depends on the amplitude. (There are several good textbooks on nonlinear mechanics; we simply introduce the subject here.)

Consider a system displaced to a position x from its equilibrium position x_0 and under a force $F(x)$. Let us expand $F(x)$ in a Taylor series about x_0; that is,

$$F(x) = F(x_0) + \left(\frac{dF}{dx}\right)_{x_0}(x - x_0) + \frac{1}{2}\left(\frac{d^2F}{dx^2}\right)_{x_0}(x - x_0)^2 + \frac{1}{6}\left(\frac{d^3F}{dx^3}\right)_{x_0}(x - x_0)^3 + \cdots \tag{4.62}$$

where $F(x_0) = 0$ because x_0 is the equilibrium point. Let $x_0 = 0$ be the origin. Define

$$\left(\frac{dF}{dx}\right)_0 = k_1, \quad \frac{1}{2}\left(\frac{d^2F}{dx^2}\right)_0 = k_2, \quad \frac{1}{6}\left(\frac{d^3F}{dx^3}\right)_0 = k_3, \quad \ldots \tag{4.63}$$

and write Eq. (4.62) as

$$F(x) = k_1x + k_2x^2 + k_3x^3 + \cdots \tag{4.64}$$

We need not carry higher terms. If we consider only those forces that lead to stable equilibrium for symmetrical systems, the even terms must vanish; that is, $k_2 = k_4 = \cdots = 0$ and

$$F(x) = k_1x + k_3x^3 \tag{4.65}$$

This force is *symmetrical* about the equilibrium $x = 0$; that is, the magnitude of the force exerted on the system is the same for x and $-x$. If we set $k_1 = -k$, where k is positive, and $k_3 = -\epsilon$, we get

$$F(x) = -kx - \epsilon x^3 \tag{4.66}$$

Remember, if $\epsilon > 0$ the system is *hard* and if $\epsilon < 0$ the system is *soft*. For this force, the corresponding symmetrical potential is given by (force $F = -dV/dx$)

$$V(x) = \tfrac{1}{2}kx^2 + \tfrac{1}{4}\epsilon x^4 \tag{4.67}$$

On the other hand, if the system is asymmetrical, the force, from Eq. (4.64), after substituting $k_1 = -k$, $k_2 = -\lambda$ and setting $k_3 = 0$, is

$$F(x) = -kx - \lambda x^2 \tag{4.68}$$

Hence the *asymmetrical potential* is

$$V(x) = \tfrac{1}{2} kx^2 + \tfrac{1}{3} \lambda x^3 \tag{4.69}$$

We shall now discuss some situations dealing with symmetrical and asymmetrical potentials.

Symmetrical Nonlinear System

Consider a mass m which is suspended between two identical strings (or springs), as shown in Fig. 4.4(a). The strings are elastic with a force constant k_0 and tied to points A and B. When this system is in position AOB it is in equilibrium and the tension in each string is S_0 as shown. Let us now displace the mass m horizontally through a distance x, as shown in Fig. 4.4(b). The change in length of each string is $(l - l_0)$; hence the restoring force is $k_0(l - l_0)$. The tension S in each string when it is in the displaced position is

$$S = S_0 + k_0(l - l_0) \tag{4.70}$$

We resolve S into components. The vertical components cancel each other, while the sum of the horizontal components is $-2S \sin \theta$. Thus the motion of the mass m is described by the equation

$$m\ddot{x} = -2S \sin \theta \tag{4.71}$$

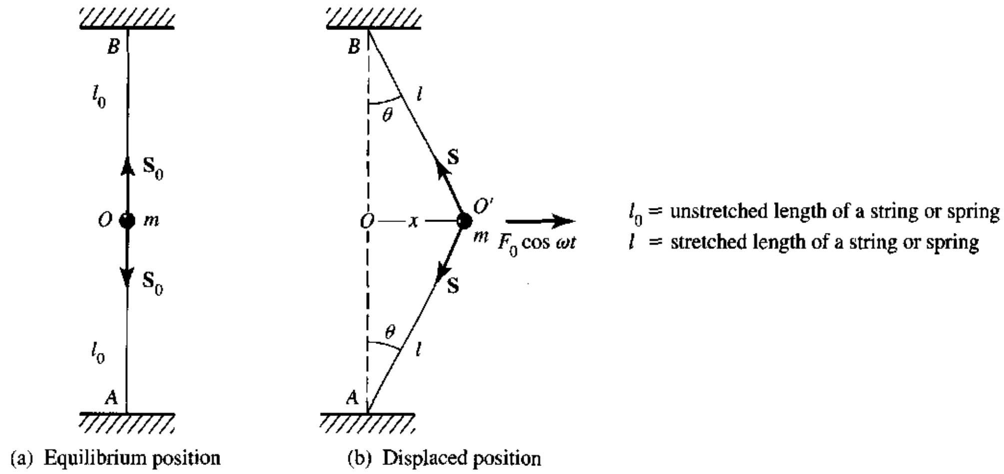


Figure 4.4 A mass m tied to two strings (or springs) constitutes a symmetrical nonlinear system: (a) in equilibrium, and (b) in a displaced position.

Remember that we have assumed no damping and also no driving force. (Later we shall apply the force $F_0 \cos \omega t$ and again solve the problem.) From Eqs. (4.70) and (4.71), we obtain

$$m\ddot{x} = -2[S_0 + k_0(l - l_0)] \sin \theta \tag{4.72}$$

From Fig. 4.4(b), we can calculate the values for l and $\sin \theta$ to be

$$l = (l_0^2 + x^2)^{1/2} = l_0\left(1 + \frac{x^2}{l_0^2}\right)^{1/2} \tag{4.73a}$$

and

$$\sin \theta = \frac{x}{l} = x(l_0^2 + x^2)^{-1/2} = \frac{x}{l_0}\left(1 + \frac{x^2}{l_0^2}\right)^{-1/2} \tag{4.73b}$$

Substituting in Eq. (4.72) yields

$$m\ddot{x} = -2\left[S_0 + k_0 l_0\left\{\left(1 + \frac{x^2}{l_0^2}\right)^{1/2} - 1\right\}\right]\frac{x}{l_0}\left(1 + \frac{x^2}{l_0^2}\right)^{-1/2} \tag{4.74}$$

Since x^2/l_0^2 is a small quantity, we can use the binomial theorem to expand

$$\left(1 + \frac{x^2}{l_0^2}\right)^{\pm 1/2} = 1 \pm \frac{1}{2}\frac{x^2}{l_0^2} + \cdots$$

Substituting these in Eq. (4.74), we get (dropping the x^5 term)

$$m\ddot{x} = -\frac{2S_0}{l_0}x - \left(\frac{k_0}{l_0^2} - \frac{S_0}{l_0^3}\right)x^3 \tag{4.75}$$

Let

$$\frac{2S_0}{l_0} = k \quad \text{and} \quad \left(\frac{k_0 l_0 - S_0}{l_0^3}\right) = \epsilon \tag{4.76}$$

Hence we can write Eq. (4.75) as

$$m\ddot{x} = -kx - \epsilon x^3 \tag{4.77}$$

which is the required equation representing a nonlinear system. Note that ϵ is a small quantity that is positive for a hard spring and negative for a soft spring. Let us assume that the approximate solution of Eq. (4.77) is still sinusoidal as in the linear systems. This should be approximately true because ϵ is a small quantity. Hence,

$$x = A \cos \omega t \tag{4.78}$$

Substituting for $x = x_1$ and $x^3 = x_1^3$ in the right side of Eq. (4.77), we get a new equation in x_1; that is,

$$m\ddot{x}_1 = -kA \cos \omega t - \epsilon A^3 \cos^3 \omega t \tag{4.79}$$

Substituting

$$\cos^3 \omega t = \tfrac{1}{4}(3 \cos \omega t + \cos 3\omega t)$$

in Eq. (4.79) and rearranging,

$$m\ddot{x}_1 = -(kA + \tfrac{3}{4}\epsilon A^3) \cos \omega t - \tfrac{1}{4}\epsilon A^3 \cos 3\omega t \tag{4.80}$$

which on integration (assuming the integration constants to be zero) gives the required solution:

$$x_1 = \frac{1}{m\omega^2}\left(kA + \frac{3}{4}\epsilon A^3\right) \cos \omega t + \frac{\epsilon A^3}{36\omega^2} \cos 3\omega t \tag{4.81}$$

This is the solution for a first-order approximation.

To find the relation between ω and A we can make use of the assumption that ϵ is small. Thus, substituting a first-order approximation $x = x_1 = A \cos \omega t$ given by Eq. (4.78) in Eq. (4.81) and dropping the last term, or by comparing terms (the second term on the right is zero), we get

$$\omega^2 = \frac{k}{m} + \frac{3}{4}\frac{\epsilon}{m}A^2 \tag{4.82}$$

which indicates that the natural frequency ω and hence the period $T = 2\pi/\omega$ are functions of the amplitude A. The quantity ω^2 increases or decreases from ω_0^2 by an amount $(3\epsilon/4m)A^2$ depending on the magnitude and the sign of ϵ.

If there were an external driving force $F = F_0 \cos \omega t$ acting on the system, as shown in Fig. 4.4(b), Eq. (4.77) would take the form

$$m\ddot{x} = -kx - \epsilon x^3 + F_0 \cos \omega t \tag{4.83}$$

Following exactly the outlined procedure, we obtain the following solution (see Problem 4.26):

$$x_1 = \frac{1}{m\omega^2}\left(kA + \frac{3}{4}\epsilon A^3 - \frac{F_0}{m}\right) \cos \omega t + \frac{\epsilon A^3}{36\omega^2} \cos 3\omega t \tag{4.84}$$

We shall not carry on with the discussion of resonances in this case; although they do occur, they are quite different from those discussed in linear systems.

Asymmetrical Nonlinear System

According to Eqs. (4.68) and (4.69), the asymmetric force and potential representing such a nonlinear system are

$$F(x) = -kx - \lambda x^2 \tag{4.68}$$

$$V(x) = \tfrac{1}{2}kx^2 + \tfrac{1}{3}\lambda x^3 \tag{4.69}$$

The differential equation describing such a system without damping is

$$m\ddot{x} + kx + \lambda x^2 = 0 \tag{4.85}$$

Dividing by m and substituting $k/m = \omega_0^2$ and $\lambda/m = \lambda_1$, we get

$$\ddot{x} + \omega_0^2 x + \lambda_1 x^2 = 0 \tag{4.86}$$

To obtain an approximate solution we use a *perturbation method,* as explained next.

If there were no nonlinear term ($\lambda_1 x^2$), the solution would have been x_0. Since λ_1 is a small quantity, the correct solution of Eq. (4.86) can be obtained by adding a small correction term to x_0; that is,

$$x(t) \simeq x_0 + \lambda_1 x_1 \tag{4.87}$$

[To have higher-order corrections we must write

$$x(t) = x_0 + \lambda_1 x_1 + \lambda_1^2 x_2 + \lambda_1^3 x_3 + \cdots]$$

Substituting Eq. (4.87) in Eq. (4.86), we get

$$(\ddot{x}_0 + \omega_0^2 x_0) + (\ddot{x}_1 + \omega_0^2 x_1 + x_0^2)\lambda_1 + 2x_0 x_1 \lambda_1^2 + x_1^2 \lambda_1^3 = 0 \tag{4.88}$$

Neglecting the higher-order terms in λ_1^2 and λ_1^3 results in

$$(\ddot{x}_0 + \omega_0^2 x_0) + (\ddot{x}_1 + \omega_0^2 x_1 + x_0^2)\lambda_1 = 0 \tag{4.89}$$

For this equation to be valid for any value of λ_1, each term must be zero; that is,

$$(\ddot{x}_0 + \omega_0^2 x_0) = 0 \tag{4.90}$$

$$(\ddot{x}_1 + \omega_0^2 x_1 + x_0^2) = 0 \tag{4.91}$$

Thus the solution can be obtained by first solving Eq. (4.90) for x_0, substituting this in Eq. (4.91), and solving for x_1. Hence the final solution will be

$$x(t) = x_0 + \lambda_1 x_1 \tag{4.92}$$

Suppose the initial conditions are such that we have the following solution for Eq. (4.90):

$$x_0 = A \sin \omega_0 t \tag{4.93}$$

Initial conditions are included in this solution; hence we need not include the transient solution. Substituting Eq. (4.93) in Eq. (4.91) yields

$$\ddot{x}_1 + \omega_0^2 x_1 = -A^2 \sin^2 \omega_0 t = -\frac{A^2}{2}(1 - \cos 2\omega_0 t) \tag{4.94}$$

The general solution of this is

$$x_1(t) = B \cos 2\omega_0 t + C \tag{4.95}$$

Substituting in Eq. (4.94) and rearranging, we get

$$\left(-\frac{A^2}{2} - 3\omega_0^2 B\right) \cos 2\omega_0 t + \left(+\frac{A^2}{2} + \omega_0^2 C\right) = 0 \tag{4.96}$$

For this to be true for any value of t we must have

$$-\frac{A^2}{2} - 3\omega_0^2 B = 0 \quad \text{or} \quad B = -\frac{A^2}{6\omega_0^2} \tag{4.97}$$

$$+\frac{A^2}{2} + \omega_0^2 C = 0 \quad \text{or} \quad C = -\frac{A^2}{2\omega_0^2} \tag{4.98}$$

Substituting in Eq. (4.95), we get

$$x_1(t) = -\frac{A^2}{6\omega_0^2}\cos 2\omega_0 t + -\frac{A^2}{2\omega_0^2} \tag{4.99}$$

Thus the general steady-state solution for a first-order approximation in λ ($\lambda_1 = \lambda/m$) is

$$x(t) \simeq x_0 + \lambda_1 x_1 = A \sin \omega_0 t + -\frac{\lambda A^2}{6m\omega_0^2}(\cos 2\omega_0 t + 3) \tag{4.100}$$

That is, the solution contains not only the free natural frequency ω_0, but also its higher harmonic $2\omega_0$. This method is not without fault. If we make the next approximation, we obtain term t (a secular term) in the solution, which is physically not acceptable for the present situation.

4.6 QUALITATIVE DISCUSSION OF MOTION AND PHASE DIAGRAMS

Energy Diagram

The following equations [Eqs. (3.2), (3.3), and (3.4)] were obtained in Chapter 3:

$$E = \tfrac{1}{2}m\dot{x}^2 + V(x) \tag{4.101}$$

$$\dot{x} = \pm\sqrt{\frac{2}{m}[E - V(x)]} \tag{4.102}$$

and

$$t_2 - t_1 = \pm\sqrt{\frac{m}{2}}\int_{x_1}^{x_2}\frac{dx}{\sqrt{E - V(x)}} \tag{4.103}$$

Once we know $V(x)$ and E, we can solve Eq. (4.103) to get a relation between x and t. But much can be learned about the qualitative nature of motion without actually solving these equations. This can be achieved in two ways: (1) by plotting $V(x)$ versus x, and (2) by plotting $\dot{x}$ versus x. Before we discuss this, we must note the following:

1. Kinetic energy cannot be negative; this will also assure that v is not imaginary.
2. Potential energy $V(x)$ cannot be greater than the total mechanical energy E of the system. If $V(x) = E$, kinetic energy K must be zero; hence the system must be at rest.

Let us consider an arbitrary potential plot $V(x)$ versus x shown by the boldface curve in Fig. 4.5(a). (A similar situation was discussed in Chapter 2, but now we discuss it by two slightly different methods.) Suppose a particle of mass m can assume different energies, as discussed here:

Particle with energy E_0: This energy corresponds to minimum potential energy V_0 and the particle is at x_0 in stable equilibrium. While at x_0 the kinetic energy is zero; and if the particle is displaced slightly it will return to x_0.

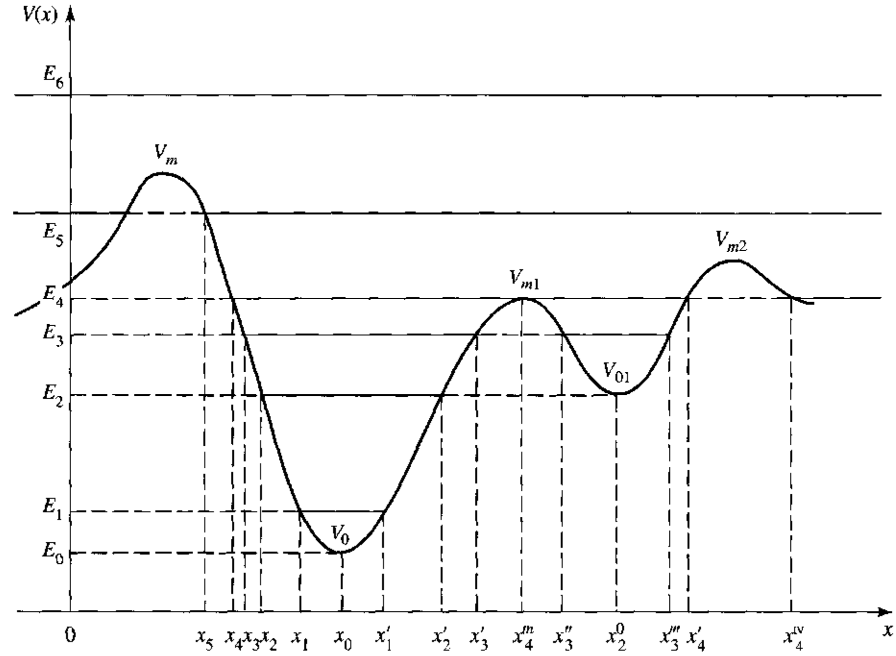
(1) *Particle with energy* E_1: This energy is greater than the minimum potential energy V_0 and the particle will oscillate between x_1 and x_1'. Since for this low energy E_1 the potential energy between x_1 and x_1' is symmetrical, the oscillations will be simple harmonic. While oscillating, the kinetic energy and the velocity are maximum when in between the points x_1 and x_1'; it has zero velocity at x_1 and x_1'. As the particle approaches x_1 or x_1', its velocity decreases, it comes to a stop, and it then reverses its direction of motion at either of the two points x_1 and x_1'. These points are called the *turning points* and the particle at these points has zero kinetic energy and maximum potential energy. The particle cannot exist in the region for which $x < x_1$ or $x > x_1'$ because this will result in an imaginary velocity.

(2) *Particle with energy* E_2: The particle can either oscillate between x_2 and x_2' or be at rest and in stable equilibrium, x_2^0. There are two turning points x_2 and x_2'. Since the potential V_{01} ($>V_0$) does not correspond to the lowest energy state (which is V_0), it is called a *metastable state.* Also, the potential $V(x)$ between x_2 and x_2' is asymmetrical; hence the oscillations are nonlinear. Again motion is not permitted in the regions $x < x_2$ and between x_2' and x_2^0.

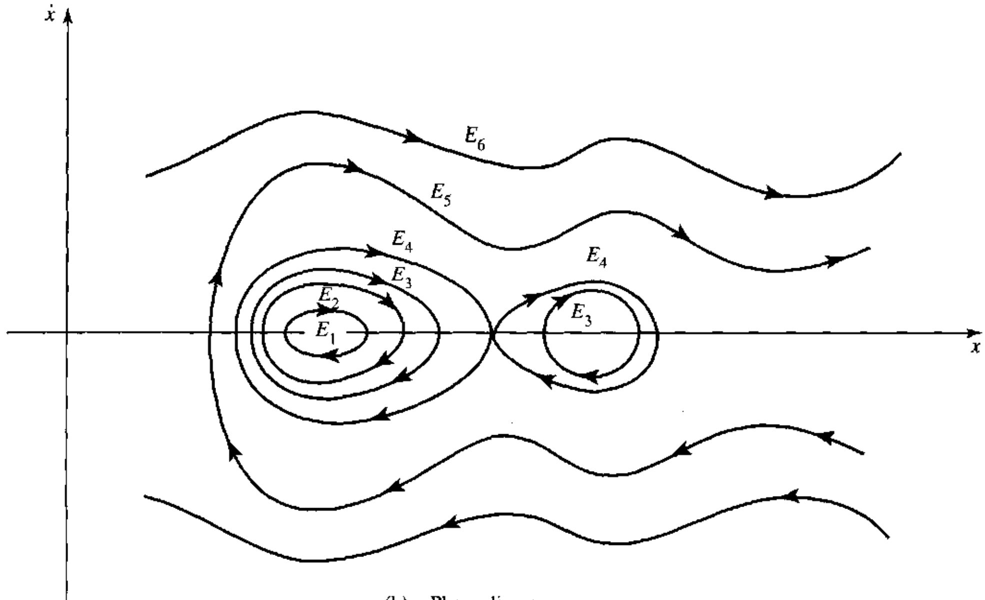
(3) *Particle with energy* E_3: There are four turning points, x_3, x_3', and x_3'', x_3'''. Because of the asymmetrical nature of the potential between x_3 and x_3', the oscillations are nonlinear in this valley. On the other hand, the potential between x_3''' is parabolic and hence symmetrical, thereby resulting in linear oscillations in this valley. Once again motion is not permitted in regions for $x < x_3$, $x_3' < x < x_3''$, and $x > x_3'''$.

(4) *Particle with energy* E_4: When the particle is at x_4^m, the potential energy is maximum (one of the maxima); hence x_4^m is a position of unstable equilibrium. If slightly displaced it could oscillate between x_4 and x_4^m or between x_4^m and x_4'; in both cases the motion is nonlinear because of asymmetrical potentials in both regions. Also, when the particle reaches x_4^m, it could move in either region. Again, for $x < x_4$ and $x > x_4'$, motion is not permitted because it results in negative kinetic energies and hence imaginary velocities.

(5) *Particle with energy* E_5: There is only one turning point, x_5. The particle traveling from the right with energy E_5 when it reaches x_5 comes to a stop, reverses its direction, and travels back to the right. While coming or going, the particle moves over hills and valleys. As it passes over the hills its velocity decreases; while passing over the valleys the velocity increases. Also, the deeper the valley, the higher the velocity is; and the higher the hill, the lower the velocity. That is, the particle will decelerate as it passes over the hills and accelerate as it passes over the valleys.



(a) Energy diagram



(b) Phase diagram

Figure 4.5 Motion of a particle with different energies E in an arbitrary potential $V(x)$ represented as (a) an energy diagram in which $V(x)$ is plotted versus x, and (b) a phase diagram in which $\dot{x}$ is plotted versus x.

(6) Particle with energy E_6: There are no turning points. The particle simply keeps on moving, slowing down over the hills and speeding up over the valleys.

The preceding discussion is the result of the energy conservation principle given by Eq. (4.101). Also, when the particle has energy E_5 and E_6, the motion of the particle is *unbound*, while in all other cases the motion is *bound.*

Phase Diagrams

To completely specify the state of motion of a one-dimensional oscillator, two quantities must be specified. According to Eq. (4.102), we have

$$\dot{x} = \pm\sqrt{\frac{2}{m}[E - V(x)]} \tag{4.102}$$

If we know $V(x)$ as a function of x, the motion may be represented by plotting $\dot{x}$ versus x. (This is in accord with a second-order differential equation in which two constant quantities are needed to describe motion.) The coordinates $\dot{x}(t)$ and $x(t)$ uniquely describe the state of motion for any time in two dimensions. Any point $P(\dot{x}, x)$ describes the state of the motion in the *phase plane*, and the locus of such points is called the *phase diagram, phase portrait*, or *phase trajectory.* In general, if we are dealing with n-dimensional motion or the system has n degrees of freedom, $2n$ coordinates will be required to describe motion in a $2n$-dimensional phase space. Also, for a constant value E, that is, for a conservative system, the motion in a phase plane is periodic, $x(t + T) = x(t)$ and $\dot{x}(t + T) = \dot{x}(t)$, and the paths are closed curves.

The phase diagram shown in Fig. 4.5(b) for potential energy $V(x)$ and for different values of E in Fig. 4.5(a) can be understood after we discuss the following.

As a first illustration, let us apply the preceding ideas to the case of a one-dimensional simple harmonic oscillator for which $V(x) = \frac{1}{2}kx^2$ Thus, for conservation of energy,

$$\tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2 = E \tag{4.104a}$$

or

$$\frac{x^2}{2E/k} + \frac{\dot{x}^2}{2E/m} = 1 \tag{4.104b}$$

which is an equation of an ellipse with $\sqrt{2E/k}$ and $\sqrt{2E/m}$ representing the semimajor and semiminor axes, respectively, and each E representing a unique ellipse. For different values of E we get a family of ellipses, as shown in Fig. 4.6. The same result can be arrived at by starting with the solution of a simple harmonic oscillator; that is,

$$x = A\cos(\omega_0 t + \phi) \tag{4.105}$$

$$\dot{x} = -\omega_0 A\sin(\omega_0 t + \phi) \tag{4.106}$$

$$\frac{x}{A} = \cos(\omega_0 t + \phi)$$

Figure 4.6

Below is the phase diagram for a one-dimensional simple harmonic oscillator for different values of E.

For given values of M, k, b, and the phase angle ϕ, we can calculate the values of $\omega 0$ and γ.

$M := 1 \qquad k := 2 \qquad b := .2$

$\omega 0 := \sqrt{\frac{k}{M}} \qquad \gamma := \frac{b}{2 \cdot M} \qquad \phi := 0$

$\omega 0 = 1.414 \qquad \gamma = 0.1$

As we know the total energy E1, E2, and E3 for three simple harmonic oscillators, we can calculate the corresponding amplitudes.

$E1 := 15 \qquad E2 := 7 \qquad E3 := 10$

$A1 := \sqrt{\frac{2 \cdot E1}{k}} \qquad A2 := \sqrt{\frac{2 \cdot E2}{k}} \qquad A3 := \sqrt{\frac{2 \cdot E3}{k}}$

$A1 = 3.873 \qquad A2 = 2.646 \qquad A3 = 3.162$

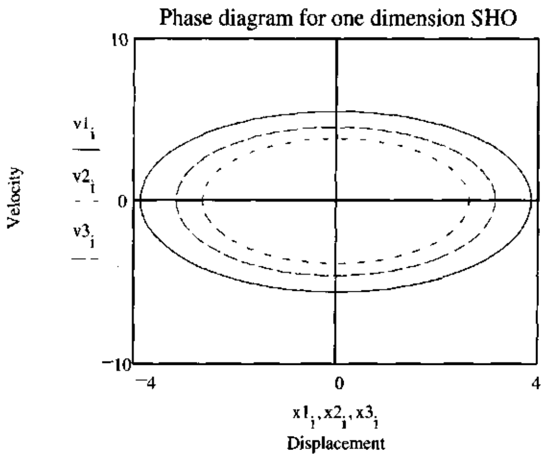
$I := 100 \qquad i := 0..I \qquad t_i := \frac{i}{10}$

With these values, we can now write the expressions for the displacements x and the corresponding velocities v (by differentiating x with respect to t) for the three harmonic oscillators and graph them.

$x1_i := A1 \cdot \cos(\omega 0 \cdot t_i + \phi) \qquad v1_i := -\omega 0 \cdot A1 \cdot \sin(\omega 0 \cdot t_i + \phi)$

$x2_i := A2 \cdot \cos(\omega 0 \cdot t_i + \phi) \qquad v2_i := -\omega 0 \cdot A2 \cdot \sin(\omega 0 \cdot t_i + \phi)$

$x3_i := A3 \cdot \cos(\omega 0 \cdot t_i + \phi) \qquad v3_i := -\omega 0 \cdot A3 \cdot \sin(\omega 0 \cdot t_i + \phi)$



(a) If the phase angle ϕ is $\pi/2$, how do the phase diagrams change?

(b) How does the increase in the values of M, k, b, and γ affect the graphs?

(c) What factors affect the change in the values of the amplitude, frequency, and energy of the oscillating system? Verify your answer by graphing.

or

$$\frac{\dot{x}}{\omega_0 A} = -\sin(\omega_0 t + \phi)$$

To eliminate t, square and add these two equations; that is,

$$\frac{x^2}{A^2} + \frac{\dot{x}^2}{\omega_0^2 A^2} = 1$$

Using $\omega_0^2 = k/m$ and $E = \frac{1}{2}kA^2$, we get

$$\frac{x^2}{2E/k} + \frac{\dot{x}^2}{2E/m} = 1$$

The first thing we note about the phase paths in Fig. 4.6 is that the motion is clockwise; the reasoning is that, for $x > 0$, $\dot{x}$ is always decreasing, while for $x < 0$, $\dot{x}$ is always increasing. Furthermore, no two phase paths cross each other. Mathematically, the reason is that each solution of the differential equation is unique. Physically, it means that if a particle is capable of changing its path and hence changing its energy, it leads to the nonconservation of energy. But we assume that for a given path a particle has a given energy.

Let us see how the phase diagram will look for an underdamped harmonic oscillator. Equations of x and $\dot{x}$ are

$$x = Ae^{-\gamma t}\cos(\omega_1 t + \phi) \tag{4.107}$$

and

$$\dot{x} = -Ae^{-\gamma t}[\gamma\cos(\omega_1 t + \phi) + \omega_1 \sin(\omega_1 t + \phi)] \tag{4.108}$$

The oscillator is continuously losing energy. Using plane polar coordinates (ρ, θ) we can show that

$$\rho = \omega_1 A e^{(-\gamma/\omega_1)\theta} \tag{4.109}$$

which is an equation of a logarithmic spiral. Without going into any details, we state that the phase path is as shown in Fig. 4.7. An oscillating particle spirals down a potential well and eventually it comes to rest when it reaches $x = 0$.

Finally, we discuss the phase diagram of a nonlinear system, as shown in Fig. 4.8(a). The asymmetric potential shown represents a hard system for $x < 0$ and a soft system for $x > 0$. Using the relation

$$\dot{x} \propto \sqrt{E - V(x)}$$

and the plot of $\dot{x}(x)$ versus x, we get the phase diagram shown in Fig. 4.8(b). The three oval-shaped closed paths correspond to three different energies, assuming that there is no damping. If damping were present, all paths would spiral down to $x = 0$.

The phase diagram in Fig. 4.5(b) should now be easy to understand. This phase diagram corresponds to the energy diagram in Fig. 4.5(a).

Figure 4.7

Below are phase diagrams for a damped harmonic oscillator using polar coordinates and rectangular coordinates.

The values given on the right are good for three damped oscillators with energies E, E1, and E2, but with the same damping constant. The phase diagram below is for the oscillator with energy E. The others can be graphed in the same way.

(**a**) How will the phase diagrams change if the energy E is increased or decreased? Check this by graphing.

(**b**) What is the fundamental difference between the two types of phase diagrams?

(**c**) Are the plots clockwise or counter-clockwise and why?

$M := .01 \quad k := 4 \quad b := .1 \quad \phi := 0$

$\omega 0 := \sqrt{\frac{k}{M}} \quad \gamma := \sqrt{\frac{b}{2 \cdot M}} \quad \omega 1 := \sqrt{\omega 0^2 - \gamma^2}$

$\omega 0 = 20 \quad \gamma = 2.236 \quad \omega 1 = 19.875$

$E := 1000 \quad E1 := 20 \quad E2 := 40$

$A := \sqrt{2 \cdot \frac{E}{k}} \quad A1 := \sqrt{2 \cdot \frac{E1}{k}} \quad A2 := \sqrt{2 \cdot \frac{E2}{k}}$

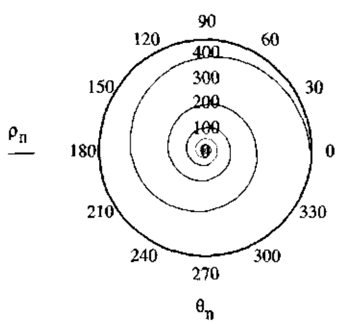
$A = 22.361 \quad A1 = 3.162 \quad A2 = 4.472$

$N := 1000 \quad n := 0..N \quad t_n := \frac{n}{100}$

Polar Graph

$\theta_n := 2 \cdot \pi \cdot \frac{n}{100}$

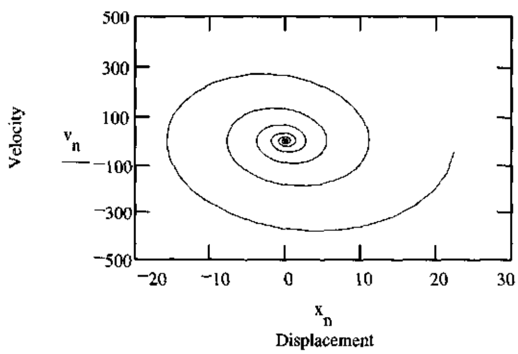
$\rho_n := \omega 1 \cdot A \cdot e^{-\frac{\gamma}{\omega 1} \cdot \theta_n}$



Rectangular Graph

$x_n := A \cdot e^{-\gamma t_n} \cdot \cos(\omega 1 \cdot t_n + \phi)$

$v_n := -A \cdot e^{-\gamma t_n} \cdot (\gamma \cos(\omega 1 \cdot t_n + \phi) + \omega 1 \cdot \sin(\omega 1 \cdot t_n + \phi))$



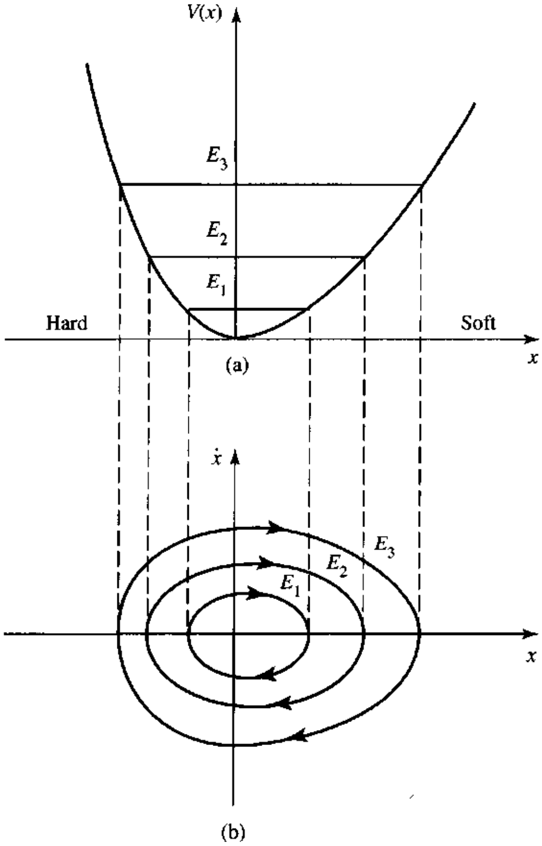


Figure 4.8 Phase and energy diagrams of a nonlinear system with asymmetrical potential.

PROBLEMS

4.1. Draw an equivalent electrical circuit for the mechanical system shown in Fig. P4.1. Set up the equations for describing motion. Calculate different possible frequencies.

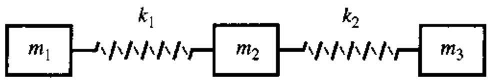


Figure P4.1

4.2. Draw an equivalent electrical circuit for the mechanical system shown in Fig. P4.2. Calculate different possible frequencies.

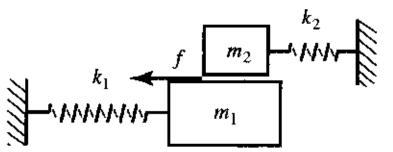


Figure P4.2

4.3. Derive Eqs. (4.17), (4.18), and (4.19).

4.4. For a spring vibrating vertically under the action of gravitational pull, draw the equivalent mechanical and electrical systems and write the necessary equations.

4.5. Discuss the electrical equivalent of the following: **(a)** average power dissipated, and **(b)** the quality factor.

4.6. Consider the electrical system shown in Fig P4.6. Calculate **(a)** the resonance frequency, **(b)** the resonance width, and **(c)** the power absorbed at resonance.

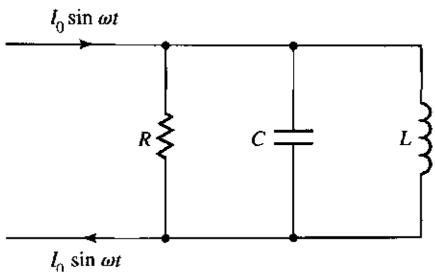


Figure P4.6

4.7. Calculate the oscillation frequency for the circuit in which $R = 150$ ohm (Ω), $C = 20$ microfarad (μF), and $L = 0.1$ henry (H).

4.8. Show that in an *RLC* circuit for which *R* is negligibly small the logarithmic decrement of oscillation is $\simeq \pi R(C/L)^{1/2}$.

4.9. A series electrical circuit contains a resistance *R*, a capacitance *C*, and an inductor *L*. An emf $\varepsilon = \varepsilon_0 \cos \omega t$ is applied to the circuit. Solve and discuss the transient and steady-state solutions. Find the steady-state expression for the current. Do this problem in analogy with a forced harmonic oscillator. Calculate the phase angle between the current and the emf. Graph the values to describe motion.

4.10. Consider the *RLC* circuit discussed in Problem 4.9 and derive an expression for the current as a function of time *t*. Show that the current decreases to zero as the frequency of the alternating emf goes to zero. Graph this.

4.11. A source of emf is connected to two impedances Z_1 and Z_2 in series. Obtain expressions for the powers dissipated in Z_1 and Z_2.

4.12. Calculate the impedance of the circuit shown in Fig. P4.12. Show that for $\omega = \omega_0 = 1/\sqrt{LC}$ the current flow is minimum. Calculate the *Q* of the circuit.

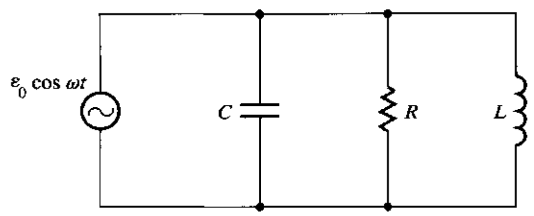


Figure P4.12

4.13. Consider the parallel *RLC* circuit shown in Fig. P4.13. Calculate *Z* from the relation

$$1/Z = 1/Z_1 + 1/Z_2$$

where Z_1 is the impedance of R_1 and C in series, while Z_2 is the impedance of R_2 and L in series: hence Z_1 and Z_2 are in parallel. Calculate the value of the total current when both current and emf are in phase. What happens when $R_1 = R_2 = 0$?

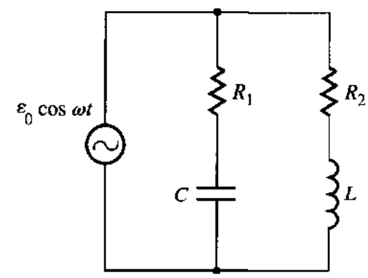


Figure P4.13

4.14. Construct the electrical analog of the system shown in Fig P4.14 and calculate the impedance. b_1 and b_2 are damping parameters resulting from the friction between each mass and surface, ω is the frequency of the driving force, and mass m_2 slides back and forth on mass m_1. Do this problem assuming the absence of spring k_2.

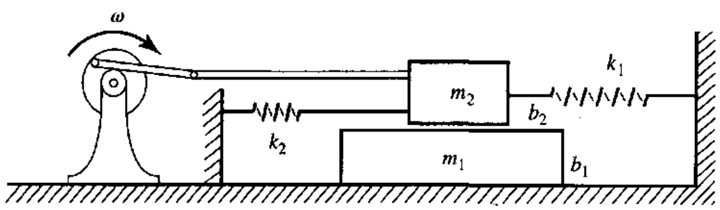


Figure P4.14

4.15. Repeat Problem 4.14 assuming the presence of both springs k_1 and k_2.

4.16. By using a Fourier technique, obtain both the sine and the cosine series in the interval $0 < \theta < \pi$ for the function

$$f(\theta) = \begin{cases} 1, & \text{for } 0 < \theta < \pi/2 \\ 0, & \text{for } \pi/2 < \theta < \pi \end{cases}$$

4.17. Calculate a Fourier series representation (Fourier transform) represented by the following function:

$$F(t) = \begin{cases} 0, & \text{if } nT < t < (n + \tfrac{1}{2})T \\ F_0 = \text{constant}, & \text{if } (n + \tfrac{1}{2})T < t < (n + 1)T \end{cases}$$

4.18. Calculate a Fourier series represenation (Fourier transform) represented by the following function:

$$F(t) = \begin{cases} 0, & \text{if } -2\pi/\omega < t < 0 \\ \sin \omega t, & \text{if } 0 < t < 2\pi/\omega \end{cases}$$

4.19. An underdamped oscillator has a natural frequency ω_0. An impulse force function of constant magnitude acts for a time $T = 2\pi/\omega_0$. Calculate the response function and give its physical interpretation by graphing it.

4.20. A linear oscillator is under the influence of the following force function:

$$F(t) = \begin{cases} 0, & \text{if } t < 0 \\ ma \sin \omega t, & \text{if } 0 < t < \pi/\omega \\ 0, & \text{if } t > \pi/\omega \end{cases}$$

Calculate the response function. Make a graph to describe the motion.

4.21. A damped oscillator is acted on by the following force function:

$$F(t) = \begin{cases} 0, & \text{if } t < 0 \\ F_0 e^{-\gamma t} \sin \omega t, & \text{if } t > 0 \end{cases}$$

Using Green's function, calculate the response function. Graph it.

4.22. Derive Eq. (4.46) for the case of a damped harmonic oscillator.

4.23. Derive Eq. (4.53) for the case of a damped harmonic oscillator.

4.24. Extend the results obtained in Eq. (4.51) to the case of multiple impulses and obtain a solution by using Green's function of the form given by Eqs. (4.59) and (4.60).

4.25. Extend the results obtained in Eq. (4.53) to the case of multiple impulses and obtain a solution by using Green's function of the form given by Eqs. (4.59) and (4.60).

4.26. Derive Eq. (4.82) and Eq. (4.84) by the procedure outlined in the text.

4.27. A particle of mass m moving in a resistive medium is acted upon by a series of impulses of force $F(t)$ at times t_1, t_2, and so on, each for a short duration Δt. Find the resulting velocity of the particle.

4.28. Show that for a nonlinear differential equation of the type $\ddot{x} + cx^2 = 0$ the superposition principle does not hold.

4.29. Suppose in Fig. 4.4 that each string must be pulled a distance d before attaching mass m. Show that under such conditions

$$V(x) \simeq \left(k \frac{d}{l}\right)x^2 + \left[\frac{k(l-d)}{4l^3}\right]x^4$$

4.30. For Eq. (4.93), assume $x_0 = A \cos \omega_0 t$ and obtain a solution similar to the one leading to Eq. (4.100).

4.31. Consider a particle described by an equation of the form

$$\ddot{x} + \omega_0^2 x - \lambda x^2 = 0$$

Obtain a correct second-order solution by using

$$x(t) = x_0 + \lambda x_1 + \lambda^2 x_2$$

4.32. Suppose in the case of a pendulum that the amplitude of the motion is not small. Show that the horizontal motion of the pendulum may be represented approximately by

$$\ddot{x} + \frac{g}{l}x - \frac{1}{2l^3}x^3 = 0$$

4.33. Consider the case of an overdamped oscillator and draw the phase diagrams for the following cases ($v_0 = \dot{x}_0$): **(a)** $v_0 > 0$, **(b)** $v_0 < 0$ and is small, and **(c)** $v_0 > 0$ and is large.

4.34. For the case of an underdamped oscillator, derive Eq. (4.109) by using $u = \omega_1 x$, $v = \gamma x + \dot{x}$, $\theta = \omega_1 t$, and Eqs. (4.107) and (4.108) in $\rho = (u^2 + v^2)^{1/2}$. Draw the phase diagrams.

4.35. A particle of mass m is under the influence of potential $V = A|x|^n$. Show that the time period for such motion is given by

$$T = \frac{2}{n}\sqrt{\frac{2\pi m}{E}}\left(\frac{E}{A}\right)^{1-n}\frac{\Gamma(1/n)}{\Gamma(1/2 + 1/n)}$$

Calculate the value of T for $n = 2$ and 3.

4.36. Construct the phase diagram for the case $n = 2$ in Problem 4.35.

4.37. Construct the phase diagram for $n = 3$ in Problem 4.35, that is, for $V = -Ax^3$.

4.38. A particle moves under the influence of a constant force F_0 when $x < 0$ and under a constant force $-F_0$ for $x > 0$. Describe the motion by constructing a phase diagram.

SUGGESTIONS FOR FURTHER READING

ARTHURS, W., and FENSTER, S. K., *Mechanics,* Chapter 6. New York: Holt, Rinehart and Winston, Inc., 1969.

BECKER, R. A., *Introduction to Theoretical Mechanics,* Chapter 7. New York: McGraw-Hill Book Co., 1954.

CHURCHILL, R. V., *Fourier Series and Boundary Value Problems.* New York: McGraw-Hill Book Co., 1941.

DAVIS, H. F., *Fourier Series and Orthogonal Functions.* Needham Heights, Mass.: Allyn & Bacon, 1963.

FRENCH, A. P., *Vibrations and Waves,* Chapter 4. New York: W. W. Norton and Co., Inc., 1971.

HAUSER, W., *Introduction to the Principles of Mechanics,* Chapter 4. Reading, Mass.: Addison-Wesley Publishing Co., 1965.

KITTEL, C., KNIGHT, W. D., and RUDERMAN, M. A., *Mechanics,* Berkeley Physics Course, Volume 1, Chapter 7. New York: McGraw-Hill Book Co., 1965.

KU, Y. H., *Analysis and Control of Nonlinear Systems.* New York: Ronald Press, 1958.

MARION, J. B., *Classical Dynamics,* 2nd ed., Chapter 5. New York: Academic Press, Inc., 1970.

MINORSKY, N., *Introduction to Nonlinear Mechanics.* Ann Arbor, Mich.: Edwards, 1947.

SYMON, K. R., *Mechanics,* 3rd, ed., Chapter 2. Reading, Mass.: Addison-Wesley Publishing Co., 1971.